PHYS 30672 MATHS METHODS Examples 2

1. If $G(x, x')$ is the Green's function for the linear operator L, what is the Green's function $\overline{G}(x, x')$ corresponding to the linear operator $\overline{L} = f(x)L$, where $f(x) \neq 0$.

2. Find the Green's function $G(x, x')$ for the operator $Ly(x) \equiv y''(x)$ in the range $0 \leq x \leq a$, where $y(0) = y(a) = 0$
(i) in the form of an eigenfunction expansion.
(ii) in the form of simple expressions for $x < x'$ and $x > x'$.

3. Find the Green's function $G(x, x')$ for the operator
$$Ly(x) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)$$
in the range $0 < x < 1$, where $y(0)$ is finite, and $y(1) = 0$, in the form as in 2.ii above.

4. The equation of motion for a particle of unit mass moving in a viscous fluid and subject to a time-dependent force $f(t)$ is
$$\frac{dv}{dt} + \beta v = f(t)\,.$$
Use the continuity method to find the Green's function for this problem. [The differential operator is non-Hermitian, so $G(t, t') \neq G(t', t)$. What is a suitable boundary condition for G?]
Use your Green's function to find $v(t)$ in the case $f(t) = f_0\, e^{-\alpha t}$, given that $v = 0$ at time $t = 0$.
Also use your $G(t, t')$ to find the Green's function that relates the particle position $x(t)$ to the applied force. What second-order differential equation does this new Green's function satisfy?

5. The time dependent Schrödinger equation can be written in the form
$$i\hbar\frac{\partial\Psi(\boldsymbol{x}, t)}{\partial t} + \frac{\hbar^2\nabla^2\Psi(\boldsymbol{x}, t)}{2m} = V(\boldsymbol{x})\Psi(\boldsymbol{x}, t) \equiv \rho(\boldsymbol{x}, t). \tag{1}$$
Note that apart from the i in the time-derivative term, this is very similar to the diffusion equation with a source term, discussed in lectures; it can be solved by the same methods. The Green's function for the Schrödinger "wave operator" is defined by
$$\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m}\right] G_0(\boldsymbol{x}, t; \boldsymbol{x}', t') = \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')\,.$$
Use the Fourier transform technique to show that the Green's function
$$G(\boldsymbol{x}, t) = G_0(\boldsymbol{x}, t; \boldsymbol{0}, 0)$$
satisfying the causal boundary condition $G(\boldsymbol{x}, t < 0) = 0$ is given by
$$G(\boldsymbol{x}, t) = -\frac{i}{\hbar}\int \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_k t)}\frac{\mathrm{d}^3k}{(2\pi)^3}$$
for $t > 0$, where $\omega_k = \dfrac{\hbar k^2}{2m}$.
For incoming particles scattering from a short range potential, one would expect
$$\Psi(\boldsymbol{x}, t) \to \Phi(\boldsymbol{x}, t) \tag{2}$$
for both $t \to -\infty$ and for $V(\boldsymbol{x}) \to 0$, where $\Phi(\boldsymbol{x}, t)$ is a known "incoming" wavefunction satisfying
$$i\hbar\frac{\partial\Phi(\boldsymbol{x}, t)}{\partial t} + \frac{\hbar^2\nabla^2\Phi(\boldsymbol{x}, t)}{2m} = 0.$$
Write down the standard Green's function solution for (1) to obtain an equation for $\Psi(\boldsymbol{x}, t)$ in terms of $G(\boldsymbol{x} - \boldsymbol{x'}, t - t')$ and $V(\boldsymbol{x})$ and show that it satisfies the boundary conditions $\Psi(\boldsymbol{x}, t) \to 0$ for both $t \to -\infty$ and $V(\boldsymbol{x}) \to 0$.
Modify this to obtain an expression for $\Psi(\boldsymbol{x}, t)$ in terms of $G(\boldsymbol{x} - \boldsymbol{x'}, t - t')$ and $\Phi(\boldsymbol{x}, t)$ which satisfies the boundary conditions (2) and is valid to first order in the potential.

6. As discussed in lectures, the Green's-function solution of
$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = f(\boldsymbol{r}, t)$$
is
$$\phi(\boldsymbol{r}, t) = \int d^3\boldsymbol{r'} \int dt'\, f(\boldsymbol{r'}, t')\,\frac{\delta(t - t' - |\boldsymbol{r} - \boldsymbol{r'}|/c)}{4\pi\,|\boldsymbol{r} - \boldsymbol{r'}|}\,.$$
For $f(\boldsymbol{r}, t) = \delta(\boldsymbol{r} - \boldsymbol{R}(t))/\epsilon_0$, which represents a unit point charge moving along the path $\boldsymbol{R}(t)$, show that this leads to the Liénard–Wiechert potential
$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r} - \boldsymbol{R}(t')| - (\boldsymbol{r} - \boldsymbol{R}(t'))\cdot\dot{\boldsymbol{R}}(t')/c}\,,$$
where $t' = t - |\boldsymbol{r} - \boldsymbol{R}(t')|/c$ is the so-called "retarded" time.
Hint: Integrate over $\boldsymbol{r'}$ first; then, for the t' integration, use the identity
$$\delta(g(t)) = \sum_i \frac{\delta(t - t_i)}{|dg/dt|}\,,$$
where the sum runs over all solutions t_i of $g(t_i) = 0$.