PHYS20672 Complex Variables and Vector Spaces: Solutions 7

65. We need to show that

$$\langle f|\hat{K}|g\rangle = \overline{\langle g|\hat{K}|f\rangle}$$

for any choice of (differentiable) functions f and g that tend to zero for $x \to \pm\infty$. We use integration by parts:

$$\begin{aligned}
\langle f|\hat{K}|g\rangle = \langle f|\hat{1}\hat{K}|g\rangle &= \int_{-\infty}^{\infty} \underbrace{\langle f|x\rangle}_{\overline{f(x)}}\underbrace{\langle x|\hat{K}|g\rangle}_{-i\mathrm{d}g/\mathrm{d}x}\,\mathrm{d}x \quad \text{(resolution of unity)} \\
&= \underbrace{\left[\overline{f(x)}\,\{-ig(x)\}\right]_{-\infty}^{\infty}}_{0,\text{ because } f,g\to 0} + \int_{-\infty}^{\infty} i\frac{\mathrm{d}\overline{f}}{\mathrm{d}x}\,g\,\mathrm{d}x \\
&= \overline{\int_{-\infty}^{\infty} \overline{g(x)}\left\{-i\frac{\mathrm{d}f}{\mathrm{d}x}\right\}} = \overline{\langle g|\hat{K}|f\rangle}.
\end{aligned}$$

$\Rightarrow$ $\hat{K}$ is Hermitian, given the boundary conditions
$\Rightarrow$ $\hat{K}$ has real eigenvalues (as for *any* Hermitan operator).

Eigenvalue equation: $\hat{K}|e_k\rangle = k|e_k\rangle$, with $k \in \mathbb{R}$. In the x representation this reads

$$\langle x|\hat{K}|e_k\rangle = -i\frac{\mathrm{d}e_k}{\mathrm{d}x} = ke_k(x)\,,$$

which has the solution $e_k(x) = Ce^{ikx}$. [We should note, however, that this eigenfunction does not obey the boundary conditions $e_k(x) \to 0$ for $x \to \pm\infty$, and that the integral of $|e_k(x)|^2 = |C|^2$ is divergent.]

66. Same eigenvalue equation as in Q.65, but different interval and boundary conditions. The solutions are again $e_k(x) = Ce^{ikx}$, but the condition $e_k(-\pi) = e_k(\pi)$ requires

$$Ce^{-ik\pi} = Ce^{ik\pi}, \quad \text{or} \quad e^{2\pi ik} = 1.$$

Thus, k can be any integer.

Normalization: $\int_{-\pi}^{\pi} |e_k(x)|^2\mathrm{d}x = 2\pi|C|^2 = 1$, $\Rightarrow |C| = 1/\sqrt{2\pi}$. Thus,

$$e_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}, \quad \text{with} \quad k \in \mathbb{Z}\,;$$

we could include an arbitrary phase factor in the solution, but there is no reason to do so.

67. First we verify the orthogonality of the eigenvectors $|e_n\rangle$ and $|e_m\rangle$ with $n \neq m$:

$$\begin{aligned}
\langle e_n|e_m\rangle &= \int_{-\pi}^{\pi} \overline{e_n(x)}\,e_m(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,\mathrm{d}x \\
&= \frac{1}{2\pi i(m-n)}\left[e^{i(m-n)x}\right]_{-\pi}^{\pi} = 0.
\end{aligned}$$

From Q.66 we also have $\langle e_n|e_n\rangle = 1$. Hence $\langle e_n|e_m\rangle = \delta_{nm}$, which we use below.

Now, if $|f\rangle = \sum_n f_n|e_n\rangle$ (with a similar expansion for $|g\rangle$), we have

(i) $\langle e_m|f\rangle = \sum_{n=-\infty}^{\infty} f_n \overbrace{\langle e_m|e_n\rangle}^{\delta_{mn}} = f_m\,;$

in the first step we used the fact that the inner product is linear in its second argument.

(ii) Insert a resolution of unity between $\langle f|$ and $|g\rangle$ and use the result of part (i):

$$\langle f|g\rangle = \langle f|\hat{1}|g\rangle = \sum_{n=-\infty}^{\infty} \underbrace{\langle f|e_n\rangle}_{\overline{f_n}} \underbrace{\langle e_n|g\rangle}_{g_n} = \sum_{n=-\infty}^{\infty} \overline{f_n}\, g_n\,.$$

(iii) $\langle f|f\rangle = \sum_{n=-\infty}^{\infty} \overline{f_n} f_n = \sum_{n=-\infty}^{\infty} |f_n|^2$ is just a special case of part (ii).

68. Briefly —

$$\begin{aligned} f(x) &= \langle x|f\rangle = \langle x|\hat{1}|f\rangle && \text{(nothing is changed by inserting } \hat{1}) \\ &= \int_{-\infty}^{\infty} \underbrace{\langle x|e_k\rangle}_{e_k(x)} \underbrace{\langle e_k|f\rangle}_{\tilde{f}(k)}\, \mathrm{d}k && \text{(resolution of unity)} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k)\, \mathrm{d}k\,, \end{aligned}$$

which is the inverse Fourier transform.

69. We note that

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u_n(x) = \langle x|\hat{K}^2|u_n\rangle \quad \text{and} \quad x^2 u_n(x) = \langle x|\hat{X}^2|u_n\rangle\,,$$

so the eigenvalue problem can be expressed in the following form that is independent of the representation:

$$\big(\hat{K}^2 + \hat{X}^2\big)|u_n\rangle = (2n+1)|u_n\rangle\,.$$

In the k representation this becomes

$$\langle e_k|\hat{K}^2|u_n\rangle + \langle e_k|\hat{X}^2|u_n\rangle = (2n+1)\langle e_k|u_n\rangle\,, \tag{1}$$

where, as shown in Lecture 23,

$$\langle e_k|u_n\rangle \equiv \tilde{u}_n(k) \quad \text{(the Fourier transform of } u_n(x)),$$

$$\langle e_k|\hat{K}^2|u_n\rangle = k^2 \tilde{u}_n(k) \quad \text{and} \quad \langle e_k|\hat{X}^2|u_n\rangle = -\frac{\mathrm{d}^2}{\mathrm{d}k^2} \tilde{u}_n(k)\,.$$

Thus, (1) is identical to

$$k^2 \tilde{u}_n - \frac{\mathrm{d}^2 \tilde{u}_n}{\mathrm{d}k^2} = (2n+1)\tilde{u}_n\,. \tag{2}$$

[Of course, this can also be obtained by Fourier transformation of the original differential equation satisfied by $u_n(x)$.]

Now, (2) is the same differential equation as

$$-\frac{\mathrm{d}^2 u_n}{\mathrm{d}x^2} + x^2 u_n = (2n+1)u_n\,;$$

all that has changed is the name of the independent variable ($x \to k$) and the name of the function ($u_n \to \tilde{u}_n$). It is claimed that the solution for a given n is unique, so u_n and $\tilde{u}_n$ can differ only by a factor: $\tilde{u}_n(s) = C u_n(s)$, where the symbol s is standing in for either x or k.

The fact that C must have unit modulus follows from Parseval's theorem, or, equivalently, by comparing results for $\langle u_n|u_n\rangle$ evaluated in the two different representations:

$$\langle u_n|u_n\rangle = \int_{-\infty}^{\infty} \langle u_n|x\rangle\langle x|u_n\rangle\,\mathrm{d}x = \int_{-\infty}^{\infty} \overline{u_n(x)}\,u_n(x)\,\mathrm{d}x \equiv \int_{-\infty}^{\infty} |u_n(s)|^2\,\mathrm{d}s$$

and

$$\langle u_n|u_n\rangle = \int_{-\infty}^{\infty} \langle u_n|e_k\rangle\langle e_k|u_n\rangle\,\mathrm{d}k = \int_{-\infty}^{\infty} |\tilde{u}_n(k)|^2\,\mathrm{d}k \equiv \int_{-\infty}^{\infty} |C|^2|u_n(s)|^2\,\mathrm{d}s\,;$$

comparison of the right-hand sides gives $|C|^2 = 1$.

70. In my solution, I've replaced x' (from the original statement of the problem) by y or z below.

(i) To prove that the scalar product $\langle f|g\rangle$ is preserved, we could follow the same procedure as in Q.69. Alternatively, we can just use the definition of $\hat{F}$ to define two new functions

$$f'(x) \equiv \langle x|\hat{F}|f\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy} f(y)\,\mathrm{d}y\,;$$
$$g'(x) \equiv \langle x|\hat{F}|g\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixz} g(z)\,\mathrm{d}z\,.$$

The inner product of these functions is

$$\langle f'|g'\rangle = \int_{-\infty}^{\infty} \overline{f'(x)}\,g'(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{ixy}\,\overline{f(y)}\,\mathrm{d}y \int_{-\infty}^{\infty} e^{-ixz} g(z)\,\mathrm{d}z\right)\mathrm{d}x\,. \qquad (3)$$

By performing the x integration *first* and making use of $\int_{-\infty}^{\infty} e^{ix(y-z)}\,\mathrm{d}x = 2\pi\delta(y-z)$, equation (3) becomes

$$\langle f'|g'\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(y)}\,g(z)\,\delta(y-z)\,\mathrm{d}z\,\mathrm{d}y = \int_{-\infty}^{\infty} \overline{f(y)}\,g(y)\,\mathrm{d}y = \langle f|g\rangle.$$

Thus, $\hat{F}$ preserves the inner product (so that $\hat{F}$ is unitary).

(ii) Starting from

$$g'(x) \equiv \langle x|\hat{F}|g\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixz} g(z)\,\mathrm{d}z\,,$$

we apply $\hat{F}$ a second time:

$$\langle x|\hat{F}^2|g\rangle = \langle x|\hat{F}|g'\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy}\underbrace{\left(\int_{-\infty}^{\infty} e^{-iyz} g(z)\,\mathrm{d}z\right)}_{g'(y)}\mathrm{d}y$$

By doing the y integral first and using $\int_{-\infty}^{\infty} e^{-iy(x+z)}\,\mathrm{d}y = 2\pi\delta(x+z)$, we get

$$\langle x|\hat{F}^2|g\rangle = \int_{-\infty}^{\infty} g(z)\,\delta(x+z)\,\mathrm{d}z = g(-x)\,.$$

By repeating this process, we obtain $\langle x|\hat{F}^3|g\rangle = g'(-x)$ and $\langle x|\hat{F}^4|g\rangle = g(x) \equiv \langle x|g\rangle$. Thus we have $\hat{F}^4 = \hat{1}$.

(iii) The eigenvalue equation for $\hat{F}$ is

$$\hat{F}|u\rangle = \omega|u\rangle,$$

where $|u\rangle \neq |0\rangle$. Applying $\hat{F}$ to each side gives

$$\hat{F}^2|u\rangle = \omega\hat{F}|u\rangle = \omega^2|u\rangle,$$

and applying $\hat{F}$ twice more gives

$$\hat{F}^4|u\rangle = \omega^4|u\rangle.$$

But $\hat{F}^4 = \hat{1}$, so the last equation is $|u\rangle = \omega^4|u\rangle$. Since $|u\rangle$ is not the zero vector $|0\rangle$, this implies that $\omega^4 = 1 \Rightarrow \omega = 1,\ i,\ -1$, or $-i$.

(iv) In the question we defined

$$|u\rangle = \tfrac{1}{4}\big(\hat{1} + \hat{F} + \hat{F}^2 + \hat{F}^3\big)|g\rangle.$$

By applying $\hat{F}$ to each side and using $\hat{F}^4 = \hat{1}$ we find

$$\hat{F}|u\rangle = \tfrac{1}{4}\big(\hat{F} + \hat{F}^2 + \hat{F}^3 + \underbrace{\hat{F}^4}_{\hat{F}^4=\hat{1}}\big)|g\rangle = |u\rangle.$$

So $|u\rangle$ is an eigenvector of $\hat{F}$ with eigenvalue 1.

(v) In principle, it's not hard to show that $\hat{P}_1^2 = \hat{P}_1$: we just square $\hat{P}_1$ and make use of the identities $\hat{F}^4 = \hat{1}$, $\hat{F}^5 = \hat{F}$ and $\hat{F}^6 = \hat{F}^2$. Explicitly,

$$\begin{aligned}
\hat{P}_1^2 &= \tfrac{1}{16}\big(\ \hat{1} + \hat{F} + \hat{F}^2 + \hat{F}^3\big)\big(\hat{1} + \hat{F} + \hat{F}^2 + \hat{F}^3\big)\\
&= \tfrac{1}{16}\big(\big[\hat{1} + \hat{F} + \hat{F}^2 + \hat{F}^3\big]\\
&\qquad +\big[\hat{F} + \hat{F}^2 + \hat{F}^3 + \hat{F}^4\big]\\
&\qquad +\big[\hat{F}^2 + \hat{F}^3 + \hat{F}^4 + \hat{F}^5\big]\\
&\qquad +\big[\hat{F}^3 + \hat{F}^4 + \hat{F}^5 + \hat{F}^6\big]\big)\\
&= \tfrac{1}{16}\big(\big[\hat{1} + \hat{F} + \hat{F}^2 + \hat{F}^3\big]\\
&\qquad +\big[\hat{F} + \hat{F}^2 + \hat{F}^3 + \hat{1}\big]\\
&\qquad +\big[\hat{F}^2 + \hat{F}^3 + \hat{1} + \hat{F}\big]\\
&\qquad +\big[\hat{F}^3 + \hat{1} + \hat{F} + \hat{F}^2\big]\big)\\
&= \tfrac{1}{16}\big([4\hat{P}_1] + [4\hat{P}_1] + [4\hat{P}_1] + [4\hat{P}_1]\big) = \hat{P}_1\,.
\end{aligned}$$

(vi) To construct $\hat{P}_\omega$ we proceed more systematically. We require $\hat{P}_\omega$ to satisfy

$$\hat{F}\hat{P}_\omega|g\rangle = \omega\hat{P}_\omega|g\rangle\,.$$

If this is to hold for any $|g\rangle$, the operators must be the same on each side,

$$\hat{F}\hat{P}_\omega = \omega\hat{P}_\omega\,. \tag{4}$$

Inspection of the projection operator that appears in parts (iv) and (v) suggests that we try a solution of the form $\hat{P}_\omega = a\hat{1} + b\hat{F} + c\hat{F}^2 + d\hat{F}^3$, where (a, b, c, d) are constants to be determined. By inserting this into (4) we find

$$a\hat{F} + b\hat{F}^2 + c\hat{F}^3 + d\hat{1} = \omega\big(a\hat{1} + b\hat{F} + c\hat{F}^2 + d\hat{F}^3\big).$$

Comparing coefficients of $\hat{F}^n$ on each side gives

$$\begin{aligned}
\text{coeff. } \hat{F}^0 &\Rightarrow \quad d = \omega a && (\hat{F}^0 \equiv \hat{1})\\
\hat{F}^3 &\Rightarrow \quad c = \omega d = \omega^2 a\\
\hat{F}^2 &\Rightarrow \quad b = \omega c = \omega^3 a\\
\hat{F}^1 &\Rightarrow \quad a = \omega b = \omega^4 a && (\text{correct, as } \omega^4 = 1).
\end{aligned}$$

So $\hat{P}_\omega = a\big(\hat{1} + \omega^3\hat{F} + \omega^2\hat{F}^2 + \omega\hat{F}^3\big)$ will do the job. To determine the coefficient a we use the fact that $\hat{P}_\omega$ is a projection operator, so that $\hat{P}_\omega^2 = \hat{P}_\omega$. A calculation very similar to the one in part (v) shows that $a = \frac{1}{4}$. Thus,

$$\hat{P}_\omega = \tfrac{1}{4}\big(\hat{1} + \omega^3\hat{F} + \omega^2\hat{F}^2 + \omega\hat{F}^3\big).$$

Finally we verify the completeness relation for the projection operators:

$$\begin{aligned}
\textstyle\sum_\omega \hat{P}_\omega = \tfrac{1}{4}\big(&\big[\hat{1} + \hat{F} + \hat{F}^2 + \hat{F}^3\big] && \text{(from } \hat{P}_1\text{)}\\
+&\big[\hat{1} - i\hat{F} - \hat{F}^2 + i\hat{F}^3\big] && \text{(from } \hat{P}_i\text{)}\\
+&\big[\hat{1} - \hat{F} + \hat{F}^2 - \hat{F}^3\big] && \text{(from } \hat{P}_{-1}\text{)}\\
+&\big[\hat{1} + i\hat{F} - \hat{F}^2 - i\hat{F}^3\big]\big) && \text{(from } \hat{P}_{-i}\text{)},
\end{aligned}$$

which cancels down to give $\sum_\omega \hat{P}_\omega = \hat{1}$.

Meaning of the completeness relation: Any function g that can be Fourier transformed can be decomposed as a sum of (up to) four eigenfunctions of the Fourier transformation operator, $\hat{F}$; i.e.,

$$\langle x|g\rangle = \langle x|\hat{1}|g\rangle = \sum_\omega \langle x|\hat{P}_\omega|g\rangle \equiv \sum_\omega \langle x|g_\omega\rangle,$$

where $|g_\omega\rangle = \hat{P}_\omega|g\rangle$ and $\hat{F}|g_\omega\rangle = \omega|g_\omega\rangle$.