PHYS20672 Complex Variables and Vector Spaces: Examples 5

Lower priority: ‡. Lowest priority: ‡‡. Harder problem, but still good practice: *.

36. Evaluate the following integrals using contour integration:

$$\text{(a)} \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x \qquad \text{(b) ‡} \int_{-\infty}^{\infty} \frac{x^4}{1+x^8}\,\mathrm{d}x \qquad \text{(c)} \int_{-\infty}^{\infty} \frac{1}{(x^2-2x+5)^2}\,\mathrm{d}x$$

37. Evaluate the following integrals using contour integration; in each case check that the conditions for Jordan's lemma to hold are satisfied:

$$\text{(a)} \int_{-\infty}^{\infty} \frac{x\sin x}{(1+x^2)^2}\,\mathrm{d}x \qquad \text{(b)} \int_{-\infty}^{\infty} \frac{\sin \pi x}{1+x+x^2}\,\mathrm{d}x$$

What would we get in each case if we replaced sin by cos?

38. Let a be a real number, and C be the (open) contour round a semicircle of radius ϵ, centred on the point $z = a$, starting and ending on the real axis and taken anticlockwise. Consider the integral around C of $(z-a)^n$ where n is an integer which can be positive, zero or negative. Show that the integral vanishes for odd n, except for $n = -1$, and is πi for $n = -1$. Show also that for even n, the limit as $\epsilon \to 0$ is zero if $n > -1$ and undefined if $n < -1$.
Hence show that if $f(z)$ has a simple pole at $z = a$, the integral around C is

$$\lim_{\epsilon\to 0} \int_C f(z)\,\mathrm{d}z = \frac{1}{2}\oint f(z)\,\mathrm{d}z = i\pi b_1^{z=a}, \qquad \text{where } b_1^{z=a} = \lim_{z\to a}(z-a)f(z)$$

is the residue of f at $z = a$. Evaluate the following, where in each case C is the small semicircle around the pole described above:

$$\text{(a)} \lim_{\epsilon\to 0}\int_C \frac{e^z}{z}\,\mathrm{d}z \qquad \text{(b)} \lim_{\epsilon\to 0}\int_C \frac{z^2-2z+1}{z+1}\,\mathrm{d}z \qquad \text{(c)} \lim_{\epsilon\to 0}\int_C \frac{1-e^z}{z^2}\,\mathrm{d}z$$

39. The following integrals involve poles on the real axis. Find the Cauchy principal value using contour integration. Where appropriate, check that the conditions for Jordan's lemma to hold are satisfied.

$$\text{(a)} ⨍_{-\infty}^{\infty} \frac{1}{(x-2)(x^2+1)}\,\mathrm{d}x \qquad \text{(b)} ⨍_{-\infty}^{\infty} \frac{e^{ix}}{(x^2-4)}\,\mathrm{d}x \qquad \text{(c)} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$$

For (c), the pole appears if you replace $\sin^2 x$ by $\frac{1}{2}(1-\cos 2x) = \frac{1}{2}\,\mathrm{Re}(1-e^{2ix})$, so it is like the example in Lecture 15 where the principal value integral arose as an intermediate step in calculating a well-defined integral.

‡‡ The integrand in (c) is analytic for all finite z, so the integral will be independent of the path taken between $-\infty$ and ∞. Use that property [and the residue theorem] to evaluate the integral *without* introducing a principal-value integral.

40. (a) Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - i\alpha}\, \mathrm{d}\omega\,,$$

where $\alpha > 0$ but t can be positive or negative. [Consider the cases of positive and negative t separately.]

(b) Evaluate

$$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{x - ia}}\, \mathrm{d}x$$

for $a > 0$ and $k < 0$.

*‡ If you like a real challenge, try the case $k > 0$. For the square root function, use the branch for which $\mathrm{Re}[\sqrt{x - ia}] > 0$. Your final result should be $I = (1+i)e^{-ka}\sqrt{2\pi/k}$.

41. Choose a suitable contour to evaluate

$$\int_{0}^{\infty} \frac{\sqrt{x}}{(x+1)^2}\, \mathrm{d}x.$$

42. Use an appropriate contour integral of the functions suggested to obtain the following sums of series:

(a) $f(z) = \dfrac{\cot z}{z^4}$, $\quad \dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \dfrac{1}{4^4} + \ldots = \dfrac{\pi^4}{90}$;

(b) $f(z) = \dfrac{1}{z^5 \cos z}$, $\quad \dfrac{1}{1^5} - \dfrac{1}{3^5} + \dfrac{1}{5^5} - \dfrac{1}{7^5} + \ldots = \dfrac{5\pi^5}{1536}$.

43. ‡ By considering a change of variable $w = 1/z$, and defining $g(w) = f(1/w)$, show that

$$\oint_C f(z)\, \mathrm{d}z = \oint_{C'} \frac{g(w)}{w^2}\, \mathrm{d}w,$$

where C' is the curve on the w plane corresponding to the curve C in the z plane, but traversed in the conventional (anticlockwise) direction. For instance if C is the circle $|z| = R$, C' is the circle $|w| = 1/R$. (Pay attention to the sign!)

Hence show that the sum of the residues of $f(z)$ within C must equal the sum of the residues of $g(w)/w^2$ within C'. Verify this explicitly for $f(z) = 1/(z^2 - 3z + 2)$ and C being the circle $|z| = R$ for $R = \frac{1}{2}$, $\frac{3}{2}$ and $\frac{5}{2}$.