PHYS20672 Complex Variables and Vector Spaces: Examples 3

Lower priority: ‡. Lowest priority: ‡‡. Harder problem, but still good practice: *.

21. In each of the following cases evaluate $\int_C f(z)\,\mathrm{d}z$ for curves C_1 and C_2 whose endpoints are $a = 1$ and $b = i$; C_1 is the path that follows the axes and passes through the origin, while C_2 is the straight line segment $y = 1 - x$.

$$\text{(a) } f(z) = \operatorname{Re} z \qquad \text{(b) } f(z) = z$$

22. Evaluate $\int_C |z|\,\mathrm{d}z$ for the curves C_1 and C_2, where the endpoints are $a = 1$ and $b = -1$, and C_1 is along the x-axis and C_2 is a semicircle of unit radius in the upper half plane.

23. Writing $z = a + Re^{i\theta}$, $0 \le \theta \le 2\pi$, and using the path $|z - a| = R$, show

$$\text{(I) } \oint_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i \quad \text{and} \qquad \text{(II) } \oint_C \frac{1}{(z-a)^n}\,\mathrm{d}z = 0 \quad \text{for integer } n > 1$$

Hence using Cauchy's theorem and partial fractions (where needed), find $\oint_C f(z)\,\mathrm{d}z$ in the following cases:

(a) $f(z) = 1/(z - i)$; $C : |z| = R$, where i) $R = 1/2$, ii) $R = 2$.

(b) $f(z) = 1/(z^2 - 3z + 2)$; $C : |z| = R$, where i) $R = 1/2$, ii) $R = 3/2$, iii) $R = 5/2$.

(c) $f(z) = (z + 1)/(z^2 - 3z + 2)$ for the same contours as (b).

(d) $f(z) = (z^2 + z + 1)/(z^3 - z^2)$ for the same contours as (a).

24. Use the appropriate Cauchy integral formula to evaluate the following, where C_1 is a circle with $|z| = 1$ and C_2 is a square with corners at $\pm 2, \pm 2 + 4i$.

$$\text{(a) } \oint_{C_1} \frac{e^{3z}}{z}\,\mathrm{d}z \qquad \text{(b) } \oint_{C_1} \frac{\cos^2(2z)}{z^2}\,\mathrm{d}z \qquad \text{(c) } \oint_{C_1} \frac{\sin^2(2z)}{z^2}\,\mathrm{d}z$$

$$\text{(d) } \oint_{C_2} \frac{z^2}{z-2i}\,\mathrm{d}z \qquad \text{(e) } \oint_{C_2} \frac{z^2}{z^2+4}\,\mathrm{d}z$$

25. Show that

$$\left|\frac{1}{z^2+1}\right| \le \frac{1}{R^2-1} \qquad \text{for } |z| = R > 1.$$

(See question 4.) Hence use the estimation lemma to show that

$$\lim_{R\to\infty} \oint \frac{1}{z^2+1}\,\mathrm{d}z = 0 \qquad \text{for the circular path } |z| = R.$$

Verify the result by using Cauchy's integral formula for the case of finite $R > 1$, then let $R \to \infty$.

26. By writing $z = e^{i\theta}$ (and hence $\mathrm{d}z = ie^{i\theta}\mathrm{d}\theta$), and using formulae such as $\cos\theta = \frac{1}{2}(z+z^{-1})$, convert the following to contour integrals around the unit circle and evaluate using the appropriate Cauchy integral formulae:

$$\text{(a)} \int_0^{2\pi} \cos^4\theta\,\mathrm{d}\theta \qquad \text{(b)} \int_0^{2\pi} \sin^6\theta\,\mathrm{d}\theta$$

$$\text{(c) ‡} \int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta, \quad \text{for integer } n \geq 0$$

$$\text{(d)} \int_0^{2\pi} \frac{\cos\theta}{4\cos\theta - 5}\,\mathrm{d}\theta \qquad \text{(e)} \int_0^{2\pi} \frac{\cos 2\theta}{3\cos\theta + 5}\,\mathrm{d}\theta$$

In (c), you should be able to express your answer as $2\pi(2n-1)!!/(2n)!!$, where, e.g., $7!! = 7 \times 5 \times 3 \times 1$ and $8!! = 8 \times 6 \times 4 \times 2$ (though it's fine to leave the answer in terms of a binomial coefficient).

27. ‡ In this question you prove the Cauchy integral formula for $f^{(n)}(a)$ by induction. Start by assuming it holds for $f^{(n-1)}(a)$, and use it in the expression

$$f^{(n)}(a) = \lim_{h\to 0} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h}$$

to show that it then holds for $f^{(n)}(a)$ as well. (This follows the proof in lectures for $f'(a)$.) But since it holds for $n = 1$, it must hold for any positive integer n. If the general case is too hard, start with $f''(a)$ as a warm-up.

[Even if you don't try this question, you should learn the Cauchy integral formula: the base case $n = 0$ is unavoidable, and the cases for $n > 0$ can be recovered from the base case by formally differentiating with respect to the parameter a. This question just provides the justification for differentiating "under the integral sign".]

28. Verify that the argument theorem holds for the function $f(z) = (2z+1)/(z^2+z-6)$ and the contour $|z| = 5/2$.