PHYS20672 Complex Variables and Vector Spaces: Examples 2

Lower priority: ‡. Lowest priority: ‡‡. Harder problem, but still good practice: *.

10. Using the definition of the derivative, differentiate the following (or show that the derivative doesn't exist):

$$\text{(a) } z^3 + z^2 \qquad \text{(b) } 1/z \quad (\text{for } z \neq 0) \qquad \text{(c) } |z|^2$$

For each case, identify $u(x, y)$ and $v(x, y)$ and determine where, if anywhere, the Cauchy–Riemann equations are satisfied.

11. Assuming the usual rules for differentiation of real functions (e.g. $\mathrm{d}(\sin x)/\mathrm{d}x = \cos x$) show that

$$\text{(a) } \frac{\mathrm{d}(\sin z)}{\mathrm{d}z} = \cos z \qquad \text{(b) } \frac{\mathrm{d}(\ln z)}{\mathrm{d}z} = \frac{1}{z}$$

In each case find the region in which the Cauchy–Riemann equations are satisfied.

12. ‡ Let $f(z) = u+iv$ and $g(z) = s+it$ be analytic functions of z. Consider the function $g(w)$ where $w = u + iv$ and write the Cauchy–Riemann equations in terms of these variables (i.e. using $\partial s/\partial u$ etc). Hence show that the function $g\big(f(z)\big)$ is an analytic function of z, by showing that the Cauchy–Riemann eqations for $\partial s/\partial x$ etc are satisfied.

13. i) Prove that if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations, they are also "harmonic", that is they satisfy Laplace's equation.

    ii) Prove that the form of the Cauchy–Riemann equations in polar coordinates, given below, follows from the Cartesian form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

    iii) ‡‡ Prove that if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations, $f = u+iv$ is a function only of z and not of $\overline{z}$. (Hint: write $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/(2i)$, and show that $\partial f/\partial\overline{z}|_z = 0$ and $\partial f/\partial z|_{\overline{z}} = \mathrm{d}f/\mathrm{d}z$.)

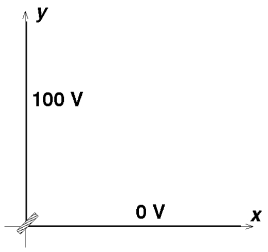
14. $u(x, y) = 2xy$ is the real part of an analytic function $f(z)$. Using the Cauchy–Riemann equations, find the conjugate function $v(x, y)$ which is the imaginary part. Hence construct $f(z)$.

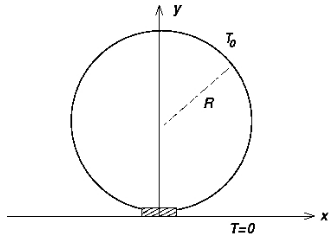
15. An analytic function $f(z)$ has imaginary part $v(x, y) = ye^x \cos y + xe^x \sin y$. Show that $v(x, y)$ is harmonic, and find the corresponding real part of $f(z)$. Express $f(z)$ in terms of z. [If you find the last step impossible to do by inspection, try substituting $x = z - iy$. If your expression for u is correct, the dependence on y should cancel out from $u + iv$, after some fairly heavy algebra.]

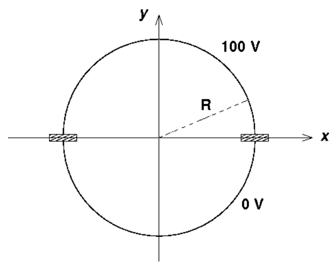
16. ‡ In general, when we change from one set of (real) coordinates x, y to another set u, v, the new element of area $\mathrm{d}u\,\mathrm{d}v$ is equal to $J\,\mathrm{d}x\,\mathrm{d}y$, where the Jacobian J is the determinant

$$J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix},$$

Show that if $f(z) = u + iv$ is analytic, $J = |\partial f/\partial x|^2 = |\mathrm{d}f/\mathrm{d}z|^2$.







The next three questions refer to the three figures above.

17. Two semi-infinite metal sheets are at right angles to each other; one (in the xz plane) is held at 0 Volts and the other (in the yz plane) is held at 100 Volts. We want to find the potential in the region $x > 0$, $y > 0$. Argue that the potential is a function of x and y only. (Henceforth z will refer to $x + iy$ as usual.)

    Use the method of conformal mapping, with the transformation $Z = \ln z$, to show that the two plates map into a parallel plate capacitor with plates at $Y = 0$ and $Y = \pi/2$. Find the potential in terms of $\{X, Y\}$ and hence in terms of $\{x, y\}$, verifying that it obeys the boundary conditions in the original geometry. Sketch some equipotentials and field lines.

18. A hot cylinder of radius R ($T = T_0$) rests on, but is insulated from, an infinite cold plate ($T = 0^{\circ}$ C). Show that the transformation $Z = R^2/z$ maps the surface of the cylinder and the plate to two infinite parallel plates at $Y = -R/2$ and $Y = 0$ respectively (take the point of contact to be $z = 0$). [This should be easy for the case $Y = 0$, but for $Y \neq 0$ you need to show that the equation $Y = \mathrm{Im}\,(R^2/z) = -R^2 y/(x^2 + y^2)$ can be rearranged as the equation of a circle in the xy plane.] Hence show that the temperature distribution in the region above the plane and outside the cylinder in the original problem is $T = 2T_0 Ry/(x^2 + y^2)$ and sketch some isotherms and lines of heat flow.

19. * Consider the conformal mapping $Z = \ln\left(\dfrac{R+z}{R-z}\right)$.

    For points with $|z| < R$, show that

    $$Y = \arctan\left(\frac{y}{R+x}\right) + \arctan\left(\frac{y}{R-x}\right) = \arctan\left(\frac{2yR}{R^2 - x^2 - y^2}\right)$$

    Find the shape of lines of constant Y in the xy plane, for $|Y| < \pi/2$.

    By taking the limit as $|z| \to R$, show that the semi-circles with $|z| = R$ above and below the x-axis correspond to $Y = \pi/2$ and $Y = -\pi/2$ respectively.

    A capacitor consists of two half-cylinders, radius R, which are insulated from one another where they nearly touch. The upper half is held at 100 V and the lower half at 0 V. Use the results above to show that the potential between the half-cylinders is

    $$\phi(x, y) = \left[50 + \frac{100}{\pi}\arctan\left(\frac{2yR}{R^2 - x^2 - y^2}\right)\right] \text{ V}.$$

    Sketch the equipotentials, and without further calculation add an educated guess at the field lines to your sketch.

20. ‡ The complex potential $w = u + iv$ for an electrostatics problem is given by $w = f(z)$. Show that the magnitude of the electric field $\mathbf{E} = -\nabla u$ is given by $|\mathbf{E}| = |\partial f/\partial x| = |\mathrm{d}f/\mathrm{d}z|$.