ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Vector Spaces for Quantum Mechanics

6 June 2012, 14:00 - 15:30

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. a) For each of the following matrices, state whether it is Hermitian, unitary, both, or neither:

$$\text{(i)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \text{(ii)} \begin{pmatrix} 2 & 1-i \\ 1+i & 0 \end{pmatrix}; \text{(iii)} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \text{(iv)} \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}.$$

[6 marks]

b) i) The space spanned by the eigenvectors of angular momentum for an electron ($s = 1/2$) with orbital angular momentum $l = 2$ has ten dimensions and can be written either as $V_a^6 \oplus V_b^4$ or as $V_c^5 \otimes V_d^2$. Which angular momenta are associated with each of the vector spaces V_a^6, V_b^4, V_c^5 and V_d^2?

ii) Give the number of dimensions required for the vector space of the wave functions of a free particle moving in three dimensions.

[6 marks]

c) A quantum system is in a definite state $|\psi\rangle$. If the energy is measured there is a one-half probability of finding it in state $|E_1\rangle$ with $E_1 = 1.0$ eV, a one-third probability of finding it in $|E_2\rangle$ with $E_2 = 1.5$ eV, and if neither of those it will be found in $|E_3\rangle$ with $E_3 = 6$ eV.

(i) What is the expectation value of the energy, $\langle E\rangle$?

(ii) Write down a possible expression for $|\psi\rangle$ in terms of $|E_1\rangle$, $|E_2\rangle$, and $|E_3\rangle$, consistent with the information given.

(iii) Say in a few words why $|\psi\rangle$ is not uniquely determined in terms of the $|E_i\rangle$ by the given information.

[7 marks]

d) Write down the notation for (i) an inner product, (ii) an outer product, and (iii) a direct product, of two vectors, using the Dirac notation. In each case state whether the relevant product is a bra, a ket, an operator, or a scalar complex number.

[6 marks]

2. The matrix representation of $\hat{J}_x$, for a particle with $j = 3/2$, in the J_z basis is

$$J_x \underset{J_z}{\longrightarrow} \hbar \begin{pmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{pmatrix}$$

In this question, work in the J_z basis throughout.

a) Write down the matrix representation for $\hat{J}_z$ for a $j = 3/2$ particle.

[4 marks]

b) Use the angular momentum commutation relations to find the $\hat{J}_y$ matrix for this system.

[7 marks]

c) Write down the eigenvalues of $\hat{J}_x$ for $j = 3/2$. There is no need to solve the eigenvalue equation for the matrix.

[4 marks]

d) Find the normalised eigenvectors of $\hat{J}_x$ for any two eigenvalues, and show that the vectors are orthogonal.

[10 marks]

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3. a) Briefly describe the concept of **entanglement** in quantum mechanics.

[6 marks]

b) Briefly describe two proposed technological applications of entanglement.

[4 marks]

c) The following state vectors describe the joint spin state of a two-particle system. In the formulae, $|\uparrow\rangle$ and $|\downarrow\rangle$ represent spin up and down along the z direction, and $|\leftarrow\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ corresponds to positive spin in the x direction:

$$\begin{aligned}
|A\rangle &= |\uparrow\rangle|\downarrow\rangle \\
|B\rangle &= \frac{|\uparrow\rangle|\downarrow\rangle - |\uparrow\rangle|\uparrow\rangle}{\sqrt{2}} \\
|C\rangle &= \frac{7}{25}|\uparrow\rangle|\uparrow\rangle + \frac{24}{25}|\downarrow\rangle|\downarrow\rangle \\
|D\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle|\leftarrow\rangle + |\leftarrow\rangle|\downarrow\rangle) - \frac{1}{2}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)
\end{aligned}$$

In which, if any, of these states are the spins entangled?

[6 marks]

d) The Hamiltonian for two non-interacting particles can be written $\hat{H}_1 \otimes \hat{I} + \hat{I} \otimes \hat{H}_2$, where $\hat{H}_1$ acts only on the first particle and $\hat{H}_2$ only on the second. Show that the time evolution operator can be written $\hat{U}_1 \otimes \hat{U}_2$, where $\hat{U}_1$ and $\hat{U}_2$ apply to the first and second particles respectively; you may assume that $\hat{U}(t) = \exp[-i\hat{H}t/\hbar]$ and is unitary.

[5 marks]

e) The state of a two-particle system, as in part (d), at $t = 0$ is

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|a\rangle|c\rangle + |b\rangle|d\rangle),$$

where the component states $|a\rangle$, $|b\rangle$ are for particle 1 and $|c\rangle$, $|d\rangle$ for particle 2 (not necessarily energy eigenstates). Write down the expression for the state at time t, $|\Psi(t)\rangle$, using the notation $U_1(t)|a\rangle = |U_1 a\rangle$ etc. Hence show that, if $|a\rangle$ and $|b\rangle$ are orthogonal, as are $|c\rangle$ and $|d\rangle$, it is impossible for $|\Psi(0)\rangle$ to evolve according to the Schrödinger equation into a separable state.

[4 marks]

4. a) A particle moving in one dimension has a wavepacket given by:

$$\langle x|\Psi\rangle \propto \exp\left[-ax^2 + \frac{ip_0 x}{\hbar}\right]$$

(i) What is the probability distribution for the particle position, x? By inspection or otherwise, write down the mean position, $\langle x\rangle$, and its uncertainty, Δx.

[6 marks]

(ii) Momentum eigenstates are represented in the x-basis as

$$\langle x|p = p'\rangle \propto \exp[ip'x/\hbar]$$

Use this fact to work out the momentum representation of the state, $\langle p|\Psi\rangle$. By inspection, identify the expection value of the momentum, $\langle p\rangle$. Hint: find and apply the unitary transformation between x and p bases. Do not worry about normalization.

[13 marks]

Note: A Gaussian of the form $N\exp[-x^2/2\sigma^2]$ *has standard deviation* σ.

b) (i) Using the fact that $\hat{p} = -i\hbar\partial/\partial x$ in the wave function representation, show that $[\hat{x}, \hat{p}] = i\hbar$.

[2 marks]

(ii) The destruction operator for a quantum harmonic oscillator is given by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

Write down $\hat{a}^\dagger$ and show that $[\hat{a}, \hat{a}^\dagger] = 1$.

[4 marks]

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

SYMBOL	DESCRIPTION	NUMERICAL VALUE
c	Velocity of light in vacuum	299 792 458 m s^{-1}, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}$ N A^{-2}, exactly
ϵ_0	Permittivity of vacuum where $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$	8.854×10^{-12} C^2 N^{-1} m^{-2}
h	Planck constant	6.626×10^{-34} J s
$\hbar$	$h/2\pi$	1.055×10^{-34} J s
G	Gravitational constant	6.674×10^{-11} m^3 kg^{-1} s^{-2}
e	Elementary charge	1.602×10^{-19} C
eV	Electronvolt	1.602×10^{-19} J
α	Fine-structure constant, $\frac{e^2}{4\pi\epsilon_0\hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	9.109×10^{-31} kg
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	9.274×10^{-24} J T^{-1}
R_∞	Rydberg energy $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha}\frac{\hbar}{m_e c}$	0.5292×10^{-10} m
Å	Angstrom	10^{-10} m
m_p	Proton mass	1.673×10^{-27} kg
$m_p c^2$	Proton rest-mass energy	938 272 MeV
$m_n c^2$	Neutron rest-mass energy	939 565 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	5.051×10^{-27} J T^{-1}
fm	Femtometre or fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12}m(^{12}C$ atom)	1.661×10^{-27} kg
N_A	Avogadro constant, atoms in gram mol	6.022×10^{23} mol^{-1}
T_t	Triple-point temperature	273.16 K, exactly
k	Boltzmann constant	1.381×10^{-23} J K^{-1}
R	Molar gas constant, $N_A k$	8.314 J mol^{-1} K^{-1}
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60}\frac{k^4}{\hbar^3 c^2}$	5.670×10^{-8} W m^{-2} K^{-4}
M_E	Mass of Earth	5.97×10^{24} kg
R_E	Mean radius of Earth	6.4×10^6 m
g	Standard acceleration of gravity	9.806 65 m s^{-2}, exactly
atm	Standard atmosphere	101 325 Pa, exactly
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.96×10^8 m
$L_\odot$	Solar luminosity	3.84×10^{26} W
$T_\odot$	Solar effective temperature	5.8×10^3 K
AU	Astronomical unit, mean Earth-Sun distance	1.496×10^{11} m
pc	Parsec	3.086×10^{16} m
	Year	3.156×10^7 s