ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Vector Spaces for Quantum Mechanics

1st June 2011, 9.45 a.m. - 11.15 a.m.

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. a) The kets $|a\rangle$ and $|b\rangle$ are represented in a certain orthonormal basis by $\begin{pmatrix} -2 \\ 2i \end{pmatrix}$ and $\begin{pmatrix} 2+3i \\ 2i \end{pmatrix}$ respectively. Find the numerical values of $\langle a|b\rangle$ and $\langle b|a\rangle$, and the norm of $|c\rangle = |a\rangle + |b\rangle$.

[6 marks]

b) For each of the following matrices, state whether it is Hermitian, unitary, both or neither:

$$\text{(i)} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}; \quad \text{(ii)} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}; \quad \text{(iii)} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}; \quad \text{(iv)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

[6 marks]

c) How many dimensions are needed for the vector space describing the combined spins of an electron and a proton (both spin-$\frac{1}{2}$)? Write down an orthonormal basis for this space in terms of the single-particle basis vectors $\{|\uparrow\rangle_\mathrm{e}, |\downarrow\rangle_\mathrm{e}\}$ and $\{|\uparrow\rangle_\mathrm{p}, |\downarrow\rangle_\mathrm{p}\}$ for the electron and proton respectively. Write down the possible eigenvalues of the total-spin operator $\hat{S}^2$ for the two-particle system. Without calculation, say which of your two-particle basis vectors are eigenvectors of $\hat{S}^2$.

[7 marks]

d) Let $\{|x\rangle\}$ be the set of eigenkets of the position operator on a function space describing one-dimensional functions of x, normalised according to the delta-function convention.

(i) Given that $|f\rangle$, $|g\rangle$ in the function space represent the functions $f(x)$, $g(x)$, write down $\langle f|g\rangle$ in terms of $f(x)$ and $g(x)$.

(ii) Write down the expansion for the identity operator $\hat{I}$ in terms of the $\{|x\rangle\}$.

(iii) Using your answer to (ii), show that $\langle f|\hat{I}|g\rangle$ is the same as your answer to (i).

[6 marks]

2. a) Given an operator $\hat{A}$, write down an expression for a matrix element A_{jk} in a particular basis, say $\{|y_i\rangle\}$. Hence, for a spin-$\frac{1}{2}$ system, find the matrix for the $\hat{S}_y$ operator, in the basis of its eigenkets $\{|+y\rangle, |-y\rangle\}$ (representing spin "up" and "down" along the y axis).

[7 marks]

b) In terms of the eigenkets of $\hat{S}_z$, i.e. $\{|\uparrow\rangle, |\downarrow\rangle\}$, we can write the $\hat{S}_y$ eigenkets as:

$$|+y\rangle = \frac{|\uparrow\rangle + i|\downarrow\rangle}{\sqrt{2}}, \quad |-y\rangle = \frac{|\uparrow\rangle - i|\downarrow\rangle}{\sqrt{2}}.$$

Construct the matrix of eigenvectors of $\hat{S}_y$ in the S_z basis, and use it to transform $|\uparrow\rangle$, $|\downarrow\rangle$, and the $\hat{S}_z$ operator from the S_z to the S_y representation. You may assume that in the S_z representation

$$\hat{S}_z \xrightarrow[S_z]{} \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[12 marks]

c) Evaluate the matrix of $\hat{A} = [\hat{S}_y, \hat{S}_z]$ using your S_y-representation matrices. What observable is represented by $\hat{A}/i\hbar$?

[6 marks]

3. a) The lowering operator $\hat{a}$ for a quantum simple harmonic oscillator (SHO) can be written:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

Show that the product $\hat{a}^\dagger\hat{a}$ is equal to the number operator $\hat{N}$, whose eigenvalues are the energy level quantum numbers n:

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega.$$

[7 marks]

b) Let $\{|n\rangle\}_{n=0}^{\infty}$ be the usual energy eigenstates of the SHO. A so-called coherent state of a quantum SHO is given by

$$|\lambda\rangle = A\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle,$$

where A is a normalising constant and λ is any complex number. Show that $|\lambda\rangle$ is an eigenstate of the lowering operator $\hat{a}$, with eigenvalue λ.

[8 marks]

c) Evaluate $\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle$ and $\langle\lambda|(\hat{a}^\dagger\hat{a})^2|\lambda\rangle$, and hence show that for state $|\lambda\rangle$, the expected value of the energy quantum number is equal to the square of its uncertainty: $\langle n\rangle = (\Delta n)^2$.

[10 marks]

You may assume that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, and that $[\hat{a}, \hat{a}^\dagger] = 1$.

4. A system has orbital angular momentum quantum number $l = 1$. In the L_z basis, $\hat{L}_y$ is represented by

$$\hat{L}_y \xrightarrow[L_z]{} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

a) The system is in a magnetic field aligned with the z-axis, and the Hamiltonian can be written $\widehat{H} = -\mu \hat{L}_z$. If the system is originally in the state with $L_y = 0$, i.e.

$$|\psi(t=0)\rangle = |L_y = 0\rangle \xrightarrow[L_z]{} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

show that the state at time t is represented in the L_z basis by

$$|\psi(t)\rangle \xrightarrow[L_z]{} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\mu t} \\ 0 \\ e^{-i\mu t} \end{pmatrix}$$

[6 marks]

b) Hence find $\langle L_y \rangle$ at time t.

[4 marks]

c) Find the other eigenvalues of $\hat{L}_y$ and their eigenvectors in the L_z basis.

[11 marks]

d) Find the probability of measuring $L_y = +\hbar$ at time t.

[4 marks]

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

SYMBOL	DESCRIPTION	NUMERICAL VALUE
c	Velocity of light in vacuum	299 792 458 m s^{-1}, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}$ N A^{-2}, exactly
ϵ_0	Permittivity of vacuum where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$	8.854×10^{-12} C^2 N^{-1} m^{-2}
h	Planck constant	6.626×10^{-34} J s
$\hbar$	$h/2\pi$	1.055×10^{-34} J s
G	Gravitational constant	6.674×10^{-11} m^3 kg^{-1} s^{-2}
e	Elementary charge	1.602×10^{-19} C
eV	Electronvolt	1.602×10^{-19} J
α	Fine-structure constant, $\frac{e^2}{4\pi\epsilon_0\hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	9.109×10^{-31} kg
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	9.274×10^{-24} J T^{-1}
R_∞	Rydberg energy $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha}\frac{\hbar}{m_e c}$	0.5292×10^{-10} m
Å	Angstrom	10^{-10} m
m_p	Proton mass	1.673×10^{-27} kg
$m_p c^2$	Proton rest-mass energy	938.272 MeV
$m_n c^2$	Neutron rest-mass energy	939.565 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	5.051×10^{-27} J T^{-1}
fm	Femtometre or fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12} m(^{12}C\text{ atom})$	1.661×10^{-27} kg
N_A	Avogadro constant, atoms in gram mol	6.022×10^{23} mol^{-1}
T_t	Triple-point temperature	273.16 K, exactly
k	Boltzmann constant	1.381×10^{-23} J K^{-1}
R	Molar gas constant, $N_A k$	8.314 J mol^{-1} K^{-1}
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60}\frac{k^4}{\hbar^3 c^2}$	5.670×10^{-8} W m^{-2} K^{-4}
M_E	Mass of Earth	5.97×10^{24} kg
R_E	Mean radius of Earth	6.4×10^6 m
g	Standard acceleration of gravity	9.806 65 m s^{-2}, exactly
atm	Standard atmosphere	101 325 Pa, exactly
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.96×10^8 m
$L_\odot$	Solar luminosity	3.84×10^{26} W
$T_\odot$	Solar effective temperature	5.8×10^3 K
AU	Astronomical unit, mean Earth-Sun distance	1.496×10^{11} m
pc	Parsec	3.086×10^{16} m
	Year	3.156×10^7 s