Random Processes — Solutions 4 PHYS 10471

4.1 Write $p = 0.99$, for success of 1 interception.

Prob. of 100 (independent!) successful interceptions

$$= p^{100} = 0.366$$

[Could regard this as a special case of the binomial distribution: prob. of k successful interceptions $= P_k = \binom{100}{k} p^k (1-p)^{100-k}$.]

If the enemy wants to have a better than 50% chance of getting a missile through, they need

$$1 - P(\text{all intercepted}) > 0.5$$

or $P(\text{all intercepted}) = p^N < 1 - 0.5 = 0.5$;

i.e. $N > \dfrac{\log 0.5}{\log p} = \dfrac{\log 0.5}{\log 0.99} = 68.97$

$\Rightarrow$ at least 69 missiles.

4.2 Prob. that gen. works $= p = 0.9$; $q = 1-p = 0.1$

prob. that k out of 5 gen. work $= P_k = \binom{5}{k} p^k q^{5-k}$ [Binomial dist.]

Hence prob. that at least 3 out of 5 work

$$= P_3 + P_4 + P_5$$

$$= \binom{5}{3} p^3 q^2 + \binom{5}{4} p^4 q^1 + p^5$$

$$= 0.0729 + 0.32805 + 0.59049$$

$$= 0.99144 \quad \text{(about 99\%)}.$$

4.3 Let $N = 32\,740\,471$.

Prob. of N_r wins in a given prize category follows a Binomial distribution

$$P_{N_r} = \binom{N}{N_r} P_r^{N_r} (1-P_r)^{N-N_r}$$

Mean: $\langle N_r \rangle = N P_r$

standard dev.: $\sigma_{N_r} = (N P_r (1-P_r))^{1/2}$

$$= (\langle N_r \rangle (1-P_r))^{1/2}.$$

Approximate values:

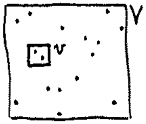
r	P_r	σ_{N_r}	$N_r - \langle N_r \rangle$
3	0.0176	753	4876
4	9.69×10^{-4}	178	−1725
5	1.84×10^{-5}	24.6	−181
6	7.15×10^{-8}	1.5	−0.3

The observed results fall many standard deviations away from the expected values (except for $r = 6$).

Assuming that the lottery is not fixed, the problem must lie in our assumption (Problem 2.3) that players choose their numbers at random: certain numbers (7?) might be favoured, and others (13?) avoided.

Exercise: [difficult!!] analyse the lottery results over a long period to discover what combinations of #s are avoided/favoured. Make sure that your own choices are unlikely to be shared with other players, assuming that you don't want to share the big prize!

4.4.



N independent trials, with probability $p = v/V$ of "success" [= "molecule in v"], and $q = 1-p$ of "failure".

It's a typical application of the binomial distribution, so

$$P_{k\,(\text{in } v)} = \binom{N}{k} p^k q^{N-k}$$

$$= \frac{N!}{k!\,(N-k)!}\, p^k (1-p)^{N-k}.$$

Mean: $\langle k \rangle = Np = Nv/V$

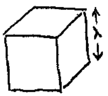
Std dev: $\sigma_k = (Npq)^{1/2} = (Nv(V-v))^{1/2}/V$

For $V \gg v$ and $\bar{n} = N/V = $ const.,

$$\langle k \rangle = \bar{n} v$$

$$\sigma_k = (\bar{n} v [1 - \tfrac{v}{V}])^{1/2} \simeq (\bar{n} v)^{1/2}.$$

We'd expect that. In this limit the binomial distribution goes over to the Poisson distribution, which has $\langle k \rangle = \sigma_k^2$.



$\lambda = 500\,\text{nm}$

$$\langle k \rangle = \sigma_k^2 = \bar{n} \lambda^3$$

[We can use the Poisson limit here because λ^3 is a tiny fraction of $V \equiv$ the whole atmosphere.]

Ideal gas law: $PV = N k_B T$ ← Boltzmann's const.

$$\Rightarrow \bar{n} = \frac{N}{V} = \frac{P}{k_B T}$$

$$= \frac{10^5}{1.4 \times 10^{-23} \times 273} = 2.6 \times 10^{25}\ \text{m}^{-3}.$$

$$\Rightarrow \langle k \rangle = \sigma_k^2 = \bar{n} \lambda^3$$

$$= 2.6 \times 10^{25} \times (5 \times 10^{-7})^3$$

$$= 3.3 \times 10^6$$

Hence typical variation in density (as a fraction of the whole) in vol λ^3 is

$$\frac{\sigma_k}{\langle k \rangle} = \frac{\langle k \rangle^{1/2}}{\langle k \rangle} = \langle k \rangle^{-1/2} \simeq \underline{5.5 \times 10^{-4}}.$$

In general,

$$\frac{\sigma_k}{\langle k \rangle} = \langle k \rangle^{-1/2} = (\bar{n} \lambda^3)^{-1/2} \propto \lambda^{-3/2}$$

which is bigger for shorter wavelengths (blue light). So blue light "sees" larger (relative) density variations than red, and is scattered more strongly. This blue-enriched scattered light gives the colour of the sky.

Similarly, light reaching us directly from the sun is red-enriched, because some of the photons of blue light have been scattered in other directions (and are seen as "blue sky" by other observers).

4.5. One-dimensional random walk, $N = 100$ steps d_i which are equally likely to be $\pm d$.

If the total displacement is x,

(a) $\langle X \rangle = \langle \sum_{i=1}^{N} d_i \rangle = \sum_{i=1}^{N} \langle d_i \rangle = 0$

(since $\langle d_i \rangle = (+d) \times \frac{1}{2} + (-d) \times \frac{1}{2} = 0$).

probabilities of step $\pm d$

(b) Mean-squared displacement

$$\langle X^2 \rangle = \langle \left(\sum_{i=1}^{N} d_i \right)^2 \rangle$$

$$= \langle \sum_{i=1}^{N} d_i^2 + \sum_{i=1}^{N-1} \left(\sum_{j>i}^{N} d_i d_j \right) \rangle$$

But $d_i^2 = d^2$, so the first term gives Nd^2; and $\langle d_i d_j \rangle = 0$, because (regardless of direction of d_i) d_j is equally likely to be in the same direction as d_i as it is to be in the opposite direction. Hence

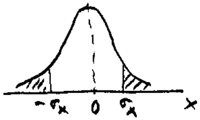
$$\langle X^2 \rangle = Nd^2 \Rightarrow x_{rms} \equiv \langle x^2 \rangle^{1/2} = \sqrt{N}\, d.$$

(c) If k is # of steps of $+d$, $X = kd + (N-k)(-d)$
$= (2k - N)d$

i.e., X and k are related linearly. So if k is approx. Gaussian-distributed for large N, so too is X. From (a) and (b) we know the mean (0) and standard deviation ($\sigma_x = x_{rms} = \sqrt{N}\, d$) for x. For $N = 100$, $\sigma_x = 10d$.

Now,

$P(X > \sigma_x) = P(X < -\sigma_x)$



$$\Rightarrow P(x > \sigma_x) = \tfrac{1}{2}(1 - P(|x| \leq \sigma_x))$$

$$= \tfrac{1}{2}(1 - 0.683) \simeq \underline{\underline{0.158}} \qquad \text{(A)}$$

[Exact value:

$$P(|X| \leq 10d) = P(x = -10d) + P(x = -8d) + \ldots + P(x = 10d) \qquad \text{(S)}$$

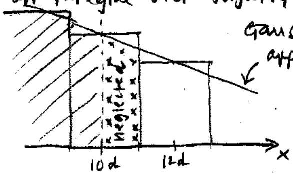
$$= P(x = 0) + 2\,(P(x = 2d) + P(x = 4d) + \cdots + P(x = 10d))$$

$$= \frac{1}{2^{100}}\binom{100}{50} + 2\left\{ \frac{1}{2^{100}}\binom{100}{51} + \frac{1}{2^{100}}\binom{100}{52} \cdots + \frac{1}{2^{100}}\binom{100}{60} \right\}$$

$$= 0.729$$

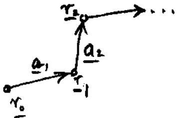
$$\Rightarrow P(x > 10d) = \tfrac{1}{2}(1 - 0.729) = 0.136.$$

The discrepancy is significant, and is due mainly to the fact that in using the Gaussian approximation we've approximated the above sum (S) by an integral over slightly too small a range:



See if you can make a simple correction to (A) to improve agreement with the exact result!]

4.6



$$(\underline{r}_N - \underline{r}_0)^2 = \left(\sum_{i=1}^{N} \underline{a}_i\right)^2$$

$$= \sum_{i=1}^{N} \underbrace{\underline{a}_i \cdot \underline{a}_i}_{a^2} + 2\sum_{i=1}^{N-1}\left(\sum_{j=i+1}^{N} \underline{a}_i \cdot \underline{a}_j\right)$$

For independent bond directions, expectation value of 2nd term is 0, so

$$R^2 = \langle(\underline{r}_N - \underline{r}_0)^2\rangle = \underline{\underline{Na^2}}.$$

If instead, $\langle \underline{a}_i \cdot \underline{a}_j \rangle = \lambda^{|i-j|} a^2$

$$R^2 = Na^2 + 2\sum_{i=1}^{N-1}\left(\sum_{j=i+1}^{N} \lambda^{|i-j|} a^2\right) \qquad \text{(P)}$$

Make a change of summation variable on the inner sum: $j = i+k$, where k runs from 1 (giving $j = i+1$) to $N-i$ (giving $j = N$).

The inner sum becomes

$$\sum_{k=1}^{N-i} \lambda^k a^2 . \qquad \text{(Q)}$$

For a sufficiently long molecule ($N \gg 1$), most of the units i are "not very close" to the end of the molecule (in the sense that λ^{N-i} is negligibly small). So we can replace the upper limit of the sum by ∞ :

$$\sum_{k=1}^{N-i} \lambda^k a^2 \doteq \lambda a^2 \underbrace{(1 + \lambda + \lambda^2 + \dots)}_{\frac{1}{1-\lambda}} = \frac{\lambda a^2}{1-\lambda}.$$

Using this approx. for (Q) in (P) gives

$$R^2 \simeq Na^2 + 2\sum_{i=1}^{N-1}\left(\frac{\lambda a^2}{1-\lambda}\right)$$

$$= Na^2 + 2(N-1)\lambda a^2/(1-\lambda)$$

neglect this in comparison with N

$$\underline{\simeq \left\{1 + \frac{2\lambda}{1-\lambda}\right\} Na^2 .}$$

<u>Optional</u>: Exact result for (Q) is

$$\sum_{k=1}^{N-i} \lambda^k a^2 = \frac{(\lambda - \lambda^{N-i+1})}{1-\lambda} a^2$$

$$\Rightarrow R^2 = Na^2 + 2\sum_{i=1}^{N-1}\left(\frac{\lambda - \lambda^{N-i+1}}{1-\lambda}\right) a^2$$

$$= Na^2 + \frac{2\lambda a^2}{1-\lambda}\left\{N-1 - \lambda^N \sum_{i=1}^{N-1} \lambda^{-i}\right\}$$

$$= \left\{1 + \frac{2\lambda}{1-\lambda}\right\}Na^2 - \frac{2\lambda a^2}{1-\lambda}\left\{1 + \lambda^N\left(\frac{\lambda^{-1} - \lambda^{-N}}{1-\lambda^{-1}}\right)\right\}$$

$$= \underbrace{\left\{1 + \frac{2\lambda}{1-\lambda}\right\}Na^2}_{\text{previous result}} - \underbrace{\frac{2\lambda a^2}{(1-\lambda)^2}\left\{1 - \lambda^N\right\}}_{\text{relatively small correction.}} \qquad \text{(R)}$$

Check: $N=1 \Rightarrow R^2 = a^2$

Expression (R) gives

$$R^2 = \left\{1 + \frac{2\lambda}{1-\lambda}\right\} a^2 - \frac{2\lambda a^2}{(1-\lambda)^2}(1-\lambda) = a^2 \checkmark$$

For $N=2$, $R^2 = 2a^2 + 2\underbrace{\langle \underline{a}_1 \cdot \underline{a}_2 \rangle}_{\lambda a^2}$

$$= 2(1+\lambda)a^2.$$

Expression (R) gives

$$R^2 = \left\{1 + \frac{2\lambda}{1-\lambda}\right\} 2a^2 - \frac{2\lambda a^2}{(1-\lambda)^2}(1-\lambda^2) \leftarrow (1-\lambda)(1+\lambda)$$

$$= \left(\frac{1+\lambda}{1-\lambda}\right)\cdot 2a^2 - 2\lambda a^2 \frac{(1+\lambda)}{1-\lambda}$$

$$= \left(\frac{1+\lambda}{1-\lambda}\right)\cdot 2a^2(1-\lambda) = 2(1+\lambda)a^2 \checkmark$$
