Random Processes : Solutions 2

PHYS 10471

2.1 prob. of a "1" $= p = \frac{1}{6}$

prob. of "not 1" $= q = \frac{5}{6}$.

prob. of "1" on k'th roll (following $k-1$ failures) $= q \times q \times q \cdots \times q \times p = q^{k-1} p$ (A)

i.e., the geometric distribution.

Expected # attempts:

$$\langle k \rangle = \sum_{k=1}^{\infty} k P_k = \sum_{k=1}^{\infty} k q^{k-1} p$$

$$= p \frac{d}{dq} \sum_{k=1}^{\infty} q^k = p \frac{d}{dq}\left(\frac{q}{1-q}\right)$$

$$= p\left(\frac{1}{1-q} + \frac{q}{(1-q)^2}\right) = \frac{p}{(1-q)^2} = \frac{1}{p} \quad \text{(B)}$$

In this case, $\langle k \rangle = 6$.

2.2 Derivation of required formulas is the same as above: results (A) and (B).

2.3 (a) The number of ways of drawing 6 balls from 49 is

$$\frac{49!}{6!(49-6)!} = \binom{49}{6}.$$

(b) The machine can select r of "my" 6 #s in $\binom{6}{r}$ ways, & $6-r$ of the other 43 #s in $\binom{43}{6-r}$ ways.

Choices are independent (they are from different sets!), so total

$$\begin{array}{l}\text{\# ways to get } r \text{ right} \\ \text{and } (6-r) \text{ wrong}\end{array} = \binom{6}{r} \times \binom{43}{6-r}.$$

(c) Prob. of getting exactly r right is just the ratio (b)/(c), assuming that all possibilities are equally likely:

$$P_r = \binom{6}{r}\binom{43}{6-r} \Big/ \binom{49}{6}$$

(d) [0] It's like 1.4 with $m = 6$, $w = 43$ and $n = 6$:

$$\binom{49}{6} = \binom{6}{0}\binom{43}{6} + \binom{6}{1}\binom{43}{5} + \cdots + \binom{6}{6}\binom{43}{0}$$

Dividing each side by $\binom{49}{6}$ and using result (c) gives $1 = \sum_{r=0}^{6} P_r$.

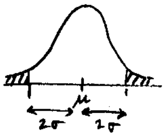
(e)

r	P_r	$\langle N_r \rangle = N P_r$
3	0.0177	577 883
4	9.69×10^{-4}	31 713
5	1.84×10^{-5}	604
6	7.15×10^{-8}	2.3

So, not bad agreement?

(f) [We'll revisit this in Examples 4.]

2.4 $\quad 25\text{ cm} = \mu + 2\sigma$



we know that prob of a result being more than 2σ from mean is 5% for a Gaussian random variable. From the symmetry about μ, $\underline{P(\text{result} > \mu + 2\sigma) = 2.5\%}$.

2.5 $\quad$ This is re-hashing 2.1/2.2. In time t, the α-ptcl. makes $k = t/\Delta t$ escape attempts. With $p = \lambda \Delta t$,

$$\langle k \rangle = \langle t/\Delta t \rangle = \frac{1}{p} = \frac{1}{\lambda \Delta t},$$

$$\Rightarrow \underline{\underline{\langle t \rangle = \frac{1}{\lambda}}}.$$

Probability of survival to time t

$$= q^{k-1} = (1-p)^{k-1}$$

$$= (1-\lambda\Delta t)^{k-1}$$

we can ignore this in comparison to k, since $(1-\lambda\Delta t) \to 1$ as $\Delta t \to 0$.

$$\simeq \left(1 - \frac{\lambda t}{k}\right)^k$$

with $k \to \infty$ as $\Delta t \to 0$. Use $\ln(1-x) = -x - \frac{x^2}{2} - \ldots$ for $x = \frac{\lambda t}{k} \ll 1$:

$$\text{prob.} = e^{k \ln(1 - \lambda t/k)} = e^{k\left(-\frac{\lambda t}{k} - \frac{\lambda^2 t^2}{2k^2} - \ldots\right)}$$

$$= e^{-\lambda t - \underbrace{\lambda^2 t^2/2k}_{\text{tends to } 0 \text{ for } k \to \infty} - \ldots} \to \underline{\underline{e^{-\lambda t}}} \quad \text{for } k \to \infty$$

2-6 $\quad$ Suppose that we have a total of N atoms today; then for each isotope,

$$N_{238} = 0.993\,N$$

$$N_{235} = 0.007\,N$$

If the numbers 1.8 Gy ago were N^0_{238} and N^0_{235}, then

$$N^0_{238} = 2^{1.8/4.5} N_{238} = 2^{1.8/4.5} \times 0.993\,N$$

$$N^0_{235} = 2^{1.8/0.7} N_{235} = 2^{1.8/0.7} \times 0.007\,N$$

[i.e., greater by factors $2^{t/t_{1/2}}$ in each case.]

$\therefore$ Proportion of ^{238}U 1.8 Gy ago was

$$\frac{N^0_{238}}{N^0_{238} + N^0_{235}} = \frac{2^{1.8/4.5} \times 0.993}{2^{1.8/4.5} \times 0.993 + 2^{1.8/0.7} \times 0.007}$$

$$= 0.969$$

$\Rightarrow$ $\underline{96.9\%\ ^{238}U \text{ and } 3.1\%\ ^{235}U}$.