PHYS 10471 Random Processes — solutions

Examples 1

1.1 (a) 37 possibilities for result of each spin

$\Rightarrow W_{all} = 37 \times 37 \times \ldots \times 37 = 37^{26}$

(b) 18 ways to get black on each spin

$\Rightarrow W_{all\,black} = 18^{26}$

(c) $\Rightarrow P(\text{all black}) = \dfrac{W_{all\,black}}{W_{all}} = \dfrac{18^{26}}{37^{26}}$ $(\simeq 7.3 \times 10^{-9})$

[of course, you can also get this directly: the spins are <u>independent trials</u> with a probability (18/37) of giving black on any trial $\Rightarrow P(\text{all black}) = \underline{\underline{(18/37)^{26}}}$.]

1.2 Total # permutations of 13 cards: $W(\text{all}) = 13!$

[i.e., 13 possibilities for the top card × 12 for the next, etc.]

(a) $W(AXX\cdots X) = 1 \times 12!$

only one possibility for top card

# of perms of remaining 12 cards.

$\Rightarrow P(AXX\cdots X) = \dfrac{W(AXX\ldots X)}{W(\text{all})} = \dfrac{12!}{13!} = \underline{\underline{\dfrac{1}{13}}}$, which

is "obvious", as there are 13 equally likely positions for the A, only one of which is the top one.

(b) A or K on top are mutually exclusive possibilities:

$W(AX\cdots X \cup KXX\cdots X) = W(AX\cdots X) + W(KX\cdots X)$

$= 12! + 12! = 2 \times 12!$

$\Rightarrow \text{Probability} = \dfrac{2 \times 12!}{13!} = \underline{\underline{\dfrac{2}{13}}}$.

(c) Same argument as for (a):

$P(X\cdots XK) = \underline{\underline{\dfrac{1}{13}}}$.

1.2(d) Here the possibilities $A\cdots$ and $\cdots K$ are <u>not</u> mutually exclusive: configurations $A\cdots K$ would be counted twice if we simply added $W(AX\cdots X)$ to $W(X\cdots XK)$; we need to "subtract out" the doubly-counted cases [so it's different from (b)] —

$W(AX\cdots X \cup X\cdots XK)$

$= W(AX\cdots X) + W(X\cdots XK) - W(AX\cdots XK)$

$= 1 \times 12! + 12! \times 1 - 1 \times 11! \times 1$

$= (12 + 12 - 1) \times 11! = 23 \times 11!$

$\Rightarrow P(AX\cdots X \cup X\cdots XK) = \dfrac{23 \times 11!}{13!} = \underline{\underline{\dfrac{23}{156}}}$.

(e) $W(AX\cdots XK) = 1 \times 11! \times 1 = 11!$

$\Rightarrow P(AX\cdots XK) = \dfrac{11!}{13!} = \underline{\underline{\dfrac{1}{156}}}$

[can't simply take (a)×(c) because the possibilities are <u>not independent</u>: once A is fixed as the top card, there are fewer places (12) where K could be.]

1.3 $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Set $x = y = 1$, then $\Rightarrow$ $\underline{2^n = \sum_{k=0}^{n} \binom{n}{k}}$.

<u>Alternatively</u>: If we are asked to select a committee, we could do it by selecting $k = n, n-1, \ldots 1$ or 0 people [the last case is no committee at all!], and these possibilities are mutually exclusive. The number of ways in total of choosing cttee is $\sum_{k=0}^{n} \binom{n}{k}$, since $\binom{n}{k}$ is the # of ways of choosing k "objects" from n, disregarding their order. But since each of ...

1·3 [continued] ... the n people is either on the cttee or not (2 possibilities for each), this total # ways must equal $2 \times 2 \ldots \times 2 = 2^n$.

Set $y = 1$: $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot 1^{n-k}$

Differentiate w.r.t. x:

$$n(x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \cdot x^{k-1}$$

then set $x = 1$ to get the result.

Alternatively: suppose we are selecting cttees of size k and nominating a chairperson from among those k. # ways for each $k = \binom{n}{k} \times k$

ways to choose k members — # of choices of chair

But we could equally well name the chair to start with (n possibilities) and then the remaining $n-1$ members (2^{n-1} possibilities — see before). Hence

$$n \times 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} \times k$$

[because of this k, it doesn't matter if we leave out the $k=0$ term!]

1·4 We'll have to suppose that $n \leq \min(M, W)$. We can select n persons out of a set of $(M+W)$ in $\binom{M+W}{n}$ ways, but the number of men selected must be one of the (mutually exclusive) possibilities $k = 0, 1, 2 \ldots n$; the # of women selected is then $n-k$.

ways to select k men $= \binom{M}{k}$

ways to select $n-k$ women $= \binom{W}{n-k}$

$$\Rightarrow \binom{M+W}{n} = \sum_{k=0}^{n} \binom{M}{k} \times \binom{W}{n-k}$$

1–5 Total # of ways to select 13 cards out of 52 is $\binom{52}{13}$ (unrestricted)

There are 32 cards no higher than 9 (Aces count high in bridge!), so there are $\binom{32}{13}$ ways to draw 13 "low cards".

Hence Probability $= \binom{32}{13} / \binom{52}{13} \simeq \underline{\underline{5.47 \times 10^{-4}}} = p$.

If we drew all $\binom{52}{13}$ hands once each, we would win £1000 on $\binom{32}{13}$ occasions and lose £1 on all the rest.

Net winnings $= £1000 \times \binom{32}{13} - £1 \times \left\{\binom{52}{13} - \binom{32}{13}\right\}$

$\Rightarrow$ Average winnings per hand

$$= \frac{\text{Net winnings}}{\binom{52}{13}} = £1000 \times p - £1 \times (1-p)$$

$$\doteq -£0.45$$

So it is a poor bet for us (about as bad as the National Lottery).

Questions 1·6 — 1·8 all use the following important result from the theory of conditional probability:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|\bar{A})\,P(\bar{A})}$$

We just need to identify the various terms from the information given in the questions.

1.6 A = "lecturer knows answer"
B = "answer is correct"

$\Big\}$ we want to find $P(A|B)$.

We've been given

$$P(A) = \frac{4}{5} \Rightarrow P(\bar{A}) = 1 - P(A) = \frac{1}{5},$$

and

$P(B|\bar{A}) = \frac{1}{3}$ (prob. that he got it right without knowing the answer)

$P(B|A) = 1$ (certain to get it right if he knows the answer!)

Hence

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

$$= \frac{1 \times \frac{4}{5}}{1 \times \frac{4}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{3 \times 4}{3 \times 4 + 1} = \underline{\underline{\frac{12}{13}}}.$$

1.7 A = "person has RA"
B = "person tests positive"

$\Big\}$ we want $P(A|B)$

We're given

$P(B/A) = 0.95$ (test is 95% effective)

$P(B/\bar{A}) = 0.01$ (1% "false positives")

and $P(A) = 0.005$ (0.5% of 20 yr-olds have RA)

Hence

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

$$= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} = 0.323$$

$$\simeq \underline{\underline{32\%}}.$$

1.8 A = "particle is pion"
B = "detector says we got a pion"

Again we want $P(A|B)$.

We are given

$P(A) = 0.9$ (beam is 90% pions)

$P(\bar{A}) = 0.1$ (beam is 10% kaons)

$P(B|A) = 0.95$ (we identify 95% of the pions correctly)

$P(B|\bar{A}) = 0.06$ (but detector incorrectly identifies 6% of kaons as pions)

Hence

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

$$= \frac{0.95 \times 0.9}{0.95 \times 0.9 + 0.06 \times 0.1}$$

$$= \underline{\underline{0.993}}$$
