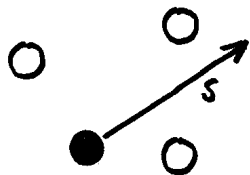


CONTINUOUS RANDOM VARS [CONTINUED]

EXAMPLE: COLLISIONS IN A GAS

We'll look at the probability that a gas molecule travels distance s without making a collision ...



... i.e., the SURVIVAL PROBABILITY $P(s)$.

Provided [as is usual!] the gas is uniform, P has one property that would help us to guess its functional form:

$$P(x+y) = P(x) P(y) \quad (\text{A})$$

[To "survive" to dist. $x+y$, it must survive to dist. x AND then survive a further dist. y !]

Strongly suggests an exponential form,

$$P(s) = e^{-s/\lambda}.$$

PHYS 10471: LECTURE 9

[COLLISIONS IN GAS]

We'll try to relate λ to microscopic parameters describing the gas: the diameter D of a molecule [assumed spherical] and the gas density [# of molecules per unit volume].

We'll need the probability of finding no molecule in a small volume ΔV . Assume that the molecules are independent of one another and that a given molecule is equally likely to be anywhere in V . The probability that a given molecule is not in ΔV is then

$$p = \frac{V - \Delta V}{V} = 1 - \frac{\Delta V}{V},$$

and the prob. that none of the N molecules is in ΔV is

$$p^N = \left(1 - \frac{\Delta V}{V}\right)^N \approx 1 - \frac{N\Delta V}{V} \quad (\text{B})$$

[using the binomial theorem for $(1+x)^N = 1 + \binom{N}{1}x + \binom{N}{2}x^2 \dots$ and assuming x is small.]

[COLLISIONS IN GAS]

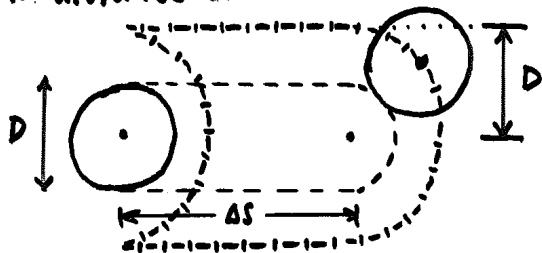
STRATEGY: Assume we know $P(s)$, and then use it to calculate $P(s+\Delta s)$



Make use of

$$P(s+\Delta s) = P(s) \times \underbrace{P(\Delta s)}_{\text{prob. of no collision in } \Delta s} \quad (\text{see } \textcircled{A})$$

But the probability of no collision in Δs is the probability that there is no molecule in the volume "swept out" in distance Δs :



$$\underline{\underline{\Delta V = \pi D^2 \Delta s}} \quad [\text{like cylinder of radius } D \text{ and height } \Delta s]$$

©

[COLLISIONS IN GAS]

CALCULATION: Now put together all ingredients to find a differential equation for $P(s)$:

$$\begin{aligned} P(s+\Delta s) &= P(s) P(\Delta s) \\ &= P(s) P(\text{no molecule in } \Delta V) \\ &\doteq P(s) \left\{ 1 - \frac{N \Delta V}{V} \right\} \quad \text{from } \textcircled{B} \\ &= P(s) - \frac{N \Delta V}{V} P(s) \end{aligned}$$

Use Taylor expansion [or definition of a derivative!] to write left-hand side as

$$P(s) + \Delta s \frac{dP(s)}{ds} + \dots, \text{ so } P(s) \text{ cancels}$$

from each side:

$$\Delta s \frac{dP}{ds} = - \frac{N \Delta V}{V} P = - \frac{\pi N D^2}{V} \Delta s P \quad (\text{using } \textcircled{C})$$

$$\text{or } \underline{\underline{\frac{dP}{ds} = - \left(\frac{\pi N D^2}{V} \right) P}}$$

We'll write $\frac{\pi N D^2}{V}$ as $\frac{1}{\lambda}$ [check that

λ — the "mean free path" — has dimensions of length!]

[COLLISIONS IN GAS]

SOLUTION:

$$\frac{dP}{ds} = -\frac{1}{\lambda} P$$

$$\Rightarrow \frac{1}{P} \frac{dP}{ds} = -\frac{1}{\lambda} \quad [\text{const.}]$$

$$\Rightarrow \int \frac{1}{P} dP = - \int \frac{1}{\lambda} ds$$

$$\text{or } \ln P = -s/\lambda + C$$

$$\Rightarrow P(s) = A e^{-s/\lambda} \quad [\text{where } A \equiv e^C]$$

Value of A from $P(0) = 1$, i.e. $A = 1$.

$$[\text{Also follows from } P(0+0) = P(0)P(0) \Rightarrow A = A^2.]$$

Hence

$$\boxed{P(s) = e^{-s/\lambda}} \quad \text{as we originally guessed.}$$

EXAMPLE 1:

[COLLISIONS IN GAS]

WARNING: Although arguments like the ones we've just made are common in the kinetic theory of gases, we've oversimplified things "slightly" by ignoring the fact that ALL the molecules are in motion — not just the one of interest.

The motion matters: a slow moving molecule will be bombarded by other, faster molecules, and so has a shorter mean free path λ than a fast molecule.

On average that effect reduces the mean free path by a factor of $\sqrt{2}$ compared with our estimate.

On the other hand, our argument is fine for scattering of a particle by fixed scattering centres [e.g., scattering of an electron by fixed impurity ions in a solid], or for diffusion of a 'light' molecule in a gas of 'heavy' [slow] ones.