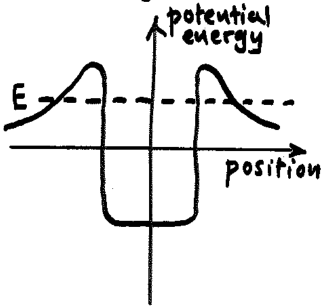
CONTINUOUS RANDOM VARS [CONTINUED]

EXAMPLE: α-DECAY OF NUCLEUS

α-particle is in a state with energy $E > 0$ inside the nucleus, but may 'tunnel' through the potential 'barrier'. What is the distribution function for the time of 'decay' [= escape of α-ptcl.]?



MODEL: Regard the α-particle as a classical particle rattling around inside the nucleus, making F collisions per second with the barrier. For each collision, the [very small] probability of escape is p.

[Compare with the problem of rolling a die until a SIX is obtained!]

SURVIVAL PROBABILITY: α-ptcl makes Ft attempts to escape in time t. Each has a probability $(1-p)$ of failure, so the prob. that the nucleus survives to time t is

$$P(t) = (1-p)^{Ft} = \left(e^{\ln(1-p)}\right)^{Ft} = \left(e^{F\ln(1-p)}\right)^{t}.$$

Write $F\ln(1-p) = -\frac{1}{\tau}$ [Note: $(1-p)<1$ so the log is <u>negative</u>.]

$$\Rightarrow \underline{\underline{P(t) = e^{-t/\tau}}}$$

But if $f(t)$ is the probability distribution for the decay times,

$$P(t) = \int_t^\infty f(t')\,dt'$$

prob. that nucleus decays at a <u>later</u> instant.

$$\Rightarrow \boxed{f(t) = -\frac{dP}{dt} = \frac{1}{\tau}e^{-t/\tau}}$$

EXPONENTIAL PROBABILITY DISTRIBUTION

CHECK: $f(t)$ should integrate to 1...

$$\int_0^\infty f(t)\,dt = \frac{1}{T}\int_0^\infty e^{-t/T}\,dt \quad (A)$$

For later use it's convenient to note that

$$I(\lambda) \equiv \int_0^\infty e^{-\lambda t}\,dt = \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda}$$

Then (A) is simply

$$\int_0^\infty f(t)\,dt = \frac{1}{T}\cdot I(1/T) = \left(\frac{1}{T}\right)/\left(\frac{1}{T}\right) = 1. \checkmark$$

MEAN TIME TO DECAY:

$$\langle t\rangle = \int_0^\infty t f(t)\,dt = \int_0^\infty t\, e^{-t/T}\,dt/T$$

Could integrate by parts, but instead we notice that

$$\int_0^\infty t\, e^{-\lambda t}\,dt = -\frac{d}{d\lambda}I(\lambda) = 1/\lambda^2$$

$$\Rightarrow \langle t\rangle = \frac{1}{(1/T)^2}\cdot\frac{1}{T} = T \quad (B)$$

i.e., T is the MEAN LIFETIME of the nucleus.

VARIANCE OF time to decay:

$$\sigma^2 = \langle t^2\rangle - \langle t\rangle^2$$

But

$$\langle t^2\rangle = \int_0^\infty t^2 f(t)\,dt = \frac{1}{T}\int_0^\infty t^2 e^{-t/T}\,dt = \frac{1}{T}\times\left\{\frac{d^2}{d\lambda^2}\int_0^\infty e^{-\lambda t}\,dt\right\}_{\lambda=1/T}$$

$$= \frac{1}{T}\left\{\frac{d^2}{d\lambda^2}I(\lambda)\right\}_{\lambda=1/T} = \frac{1}{T}\left\{\frac{2}{\lambda^3}\right\}_{\lambda=\frac{1}{T}} = 2T^2$$

$$\Rightarrow \sigma^2 = (2T^2) - (T)^2 = T^2 \quad \text{[using (B)]}$$

EXAMPLE 1: the mean lifetime of a radioactive nucleus is 10 days. What is its half-life?

SOLUTION: Recall that half-life is defined by $N(t_{1/2})/N(0) = \frac{1}{2}$, or $P(t_{1/2}) = e^{-t_{1/2}/T} = \frac{1}{2}$.

$$\Rightarrow \boxed{t_{1/2} = T\ln 2} \simeq 6.9 \text{ days}$$