EXAMPLES 1, Q3, 4

Typical "combinatorial" argument for an identity involving binomial coefficients:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \text{ [to be proved] } \textcircled{R}$$

Could proceed purely algebraically [using BINOMIAL THEOREM], but the aim is to understand the result!

$\binom{n}{k}$ = # of ways of selecting k people [a committee?] from group of n.

So $\sum_{k=0}^{n} \binom{n}{k}$ = # of ways of selecting a "committee" [of any size — including $k=0 \equiv$ no committee!]

Note that we could choose a committee by considering each of the n people in turn and deciding whether or not each is in it:

2 possibilities for each person [independent of any other person]

$\Rightarrow$ Total of $2 \times 2 \times \ldots \times 2 = 2^n$ ways

These are just different ways of calculating the same thing, so Ⓡ is proved.

STATISTICAL INDEPENDENCE

Expresses the idea that two events may have no influence on one another.

Prototype: Physically UNCONNECTED events

EXAMPLE: A die is rolled and a coin is flipped. What is the chance of getting (6, H)?

The die and coin are unconnected [in different rooms?], so in N trials we expect about N/6 to give a SIX, with H in about half of these N/6 cases:

$$\Rightarrow P(6,H) = P(6 \cap H) = \frac{1}{12} = P(6)P(H).$$

[Alternative, a priori, argument: (6, H) is just one of $6 \times 2 = 12$ possible outcomes. Assuming equal probability for each $\Rightarrow P(6 \cap H) = 1/12$.]

GENERALLY: Events A and B are said to be statistically independent if [and only if]

$$\boxed{P(A \cap B) = P(A)P(B).}$$

[STATISTICAL INDEPENDENCE, CONTINUED]

NOTE: Although our "prototype" system consists of physically unconnected parts, statistical independence doesn't REQUIRE this!

EXAMPLE: Roll a die. What is the probability that result is ODD [$O = \{1, 3, 5\}$] and divisible by THREE? [$T = \{3, 6\}$]

A priori argument: only 1 of the six, equally likely outcomes is favourable, i.e. $O \cap T = \{3\}$, so $P(O \cap T) = 1/6$.

But $P(O) = 3/6 = 1/2$ and $P(T) = 2/6 = 1/3$, so $P(O \cap T) = P(O)P(T)$ here: the events O and T are STATISTICALLY INDEPENDENT, yet only ONE die is involved.

TERMINOLOGY: Statistically independent events are also said to be UNCORRELATED.

CONDITIONAL PROBABILITY

Useful when 'events' may depend upon one another: E.g., whether or not an electrical component is likely to be faulty may depend on which manufacturer it came from...

DEFINITION:

$$P(A \text{ given } B) \equiv P(A|B) = P(A \cap B)/P(B)$$

EXAMPLE [AND MOTIVATION]: A box contains N components, of which $N(B)$ are from Birmingham [and $N(W) = N - N(B)$ from Warrington]. A subset A of the components is faulty, so $A \cap B$ are the faulty ones from B'ham.

The probability of a B'ham component being faulty is

$$\frac{N(A \cap B)}{N(B)} = \frac{N(A \cap B)/N}{N(B)/N} = \frac{P(A \cap B)}{P(B)}.$$

So if we pick a component at random from the box and discover that it's from B'ham, its chance of being faulty is

$$P(A|B) = P(A \cap B)/P(B).$$

[CONDITIONAL PROB., CONTD]

Alternative form:

$$P(A \cap B) = P(A|B)\,P(B)$$
$$= P(B|A)\,P(A) \quad [\text{since } A \cap B = B \cap A]$$

Rearranging the last equation gives

$$\boxed{P(B|A) = P(A|B)\,P(B)/P(A)} \quad \text{BAYES' THEOREM.}$$

BAYES' THEOREM reverses the rôles of A and B: it's crucial in assessing likelihood of a hypothesis being true, given some data. E.g., if

A ≡ 'testing positive for a disease'

B ≡ 'having the disease'

$P(A|B)$ ≡ probability of positive test result from person with the disease

$\neq 1$, as tests sometimes give wrong results!

But an individual who has tested positive would be more interested in $P(B|A)$: the probability that they actually <u>have</u> the disease, given their test result.

[BAYES THEOREM, CONTD]

EXAMPLE: Faulty components again.

Box contains: 2/3 Birmingham components, ≡ $P(B)$
1/3 Warrington " . ≡ $P(W)$

$P(A|B) \rightarrow$ 10% of B'ham components are faulty

$P(A) \rightarrow$ 12% of <u>all</u> components in box are faulty [faulty ≡ "Awful"]

What is the probability that a faulty component is from B'ham?

Answer: $P(B|A) = \underbrace{P(A|B)}_{10\%}\ \underbrace{P(B)}_{2/3} / \underbrace{P(A)}_{12\%} \simeq 56\%$

What's the chance that a faulty one came from Warrington?

Answer: $P(W|A) = 1 - P(B|A) = 0.44 = 44\%$

What proportion of the Warrington components are faulty?

Answer: $P(A|W) = P(W|A)\,\frac{P(A)}{P(W)} = 0.44 \times \frac{0.12}{(1/3)} = 16\%$

[NOTE the "law of total probability" is often useful here: $P(X) = P(X \cap Y) + P(X \cap \bar{Y})$
$= P(X|Y)P(Y) + P(X|\bar{Y})P(\bar{Y})$.]

[See Examples 1.6 to 1.8]