RANDOM WALK IN 1 DIMENSION

MODEL: N steps of ± 1 along x-axis, with $P(+1) = P(-1) = \frac{1}{2}$.

Probability of k steps to right and $N-k$ to left is

$$P_k(N;\tfrac{1}{2}) = \binom{N}{k} \left\{\tfrac{1}{2}\right\}^N \quad \text{BINOMIAL DISTRIB}^{N}$$

Displacement [from starting point] is related to k by

$$x = \underbrace{k}_{\text{to right}} - \underbrace{(N-k)}_{\text{to left}} = 2k - N$$

Expectation value [obvious by symmetry]:

$$\langle x \rangle = \langle 2k - N \rangle$$
$$= 2\langle k \rangle - N = 2 \times \frac{N}{2} - N = 0.$$

Variance of x:

$$\mathrm{var}(x) = \langle x^2 \rangle = \langle (2k-N)^2 \rangle$$
$$= 4 \,\mathrm{var}\, k = 4 \times N \times \tfrac{1}{2} \times \tfrac{1}{2}$$
$$= N.$$

$$\Rightarrow \sigma_x = x_{RMS} = \sqrt{N}.$$

[RANDOM WALK IN 1D]

The probability $P(x;N)$ that the walk has reached x after N steps is [of course] the same thing as $P_k(N;1/2)$ with $k = (x+N)/2$.

From the fact that P_k is Gaussian for large N it follows that $P(x;N)$ is also Gaussian in this limit:

$$\boxed{P(x;N) \simeq \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)^{1/2}} = \frac{e^{-x^2/2N}}{(2\pi N)^{1/2}}}$$

Valid for $N \gg 1$ and $x \ll N$.

"GOING NOWHERE" IN 1D

A random walk in 1D is certain to return to its starting point — eventually!

PROOF relies on the symmetry of $P(x;N)$ about $x=0$ and the fact that the probability of remaining in the range $[-L,L]$ tends to zero for $N\to\infty$:

$$P(-L\leq x\leq L;N) = \sum_{x=-L}^{L} P(x;N)$$

$$\simeq \sum_{x=-L}^{L} \frac{e^{-x^2/2N}}{\sqrt{2\pi N}} \qquad \text{GAUSSIAN LIMIT, } N\gg 1.$$

$$< \frac{2L+1}{\sqrt{2\pi N}} \longrightarrow 0 \text{ as } N\to\infty.$$

Hence the particle is certain to exit from any region, and if it started from the middle, the probability of exiting to left or right is $1/2$ [by symmetry].

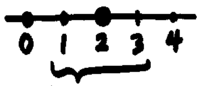
["GOING NOWHERE" PROOF]

STRATEGY: We show that the prob. of NOT returning to $x=0$ can be broken down to a sequence of independent exit events whose prob. tends to 0.

STEP 1: The particle is at $x=1$ [or -1] after the first rand. step away from $x=0$. On its next step, it either returns to $x=0$ or "exits" to $x=2$.

After STEP 1, the probability that it's at $x=2$ is $1/2$.

STEP 2:



Particle starts at centre of this region.

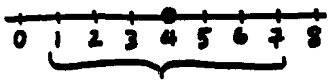
The particle has room to move around, but will eventually exit to $x=0$ or $x=4$. The prob. after STEP 2 that it has not yet returned is $\frac{1}{2}\times\frac{1}{2}$

prob. it hadn't returned to 0 after Step 1

prob. it exits to right in STEP2

["GOING NOWHERE" PROOF]

STEP 3 :



particle starts at centre of this region.

Eventually it exits, either to $x=0$ or $x=8$. The prob. after STEP 3 that it has not yet returned to $x=0$ is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$.

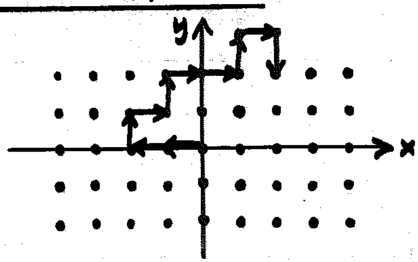
AND SO ON. After step N, the prob. that that the particle has <u>not</u> returned to $x=0$ will be $\left(\frac{1}{2}\right)^N \longrightarrow 0$ for $N \to \infty$.

Hence the probability of return to the starting point $= 1 - \frac{1}{2^N} \longrightarrow 1$.

[Aside: A similar result holds for random walks in 2D.

There is no such result for 3D: the probability of return is non-zero, but less than 1.]

RANDOM WALK IN 2D



Each step is $\uparrow, \downarrow, \rightarrow, \leftarrow$ with equal probability.

We want to find $P(x,y;N)$, the prob. that we are at (x,y) after N steps.

Break this up: prob. that M steps are $\uparrow$ or $\downarrow$ and $N-M$ are $\leftarrow$ or $\rightarrow$ is $P_M(N;\frac{1}{2})$.

Given M steps $\updownarrow$, the probability of reaching y is $P(y;M)$, since each of the steps is equally likely to be $\uparrow$ or $\downarrow$ [so it's a random walk of M steps in y].

Similarly, the prob. of reaching x in $N-M$ steps is $P(x;N-M)$.

[2D RANDOM WALK, CONTD]

Hence

$$P(x,y;N) = \sum_{M=0}^{N} P(x;N-M)\,P(y;M)\,P_M(N,\tfrac{1}{2})$$

[we sum over the mutually exclusive possibilities for M.]

Looks messy, but simplifies for $N \gg 1$, because $P_M(N,\frac{1}{2})$ is sharply peaked around $M = N/2$, with width $\propto \sqrt{N} \ll N$.

$P(x;N-M)\,P(y;M)$ varies very little over this range $\langle M\rangle \pm \sqrt{N}$, so can treat that as a [nearly!] constant factor in the sum:

$$P(x,y;N) \simeq P(x;N-\underbrace{\langle M\rangle}_{\frac{N}{2}})\,P(y;\underbrace{\langle M\rangle}_{\frac{N}{2}})\,\underbrace{\sum_M P_M(N;\tfrac{1}{2})}_{1}$$

$$= P(x;\tfrac{N}{2})\,P(y;\tfrac{N}{2})$$

$$\simeq \frac{e^{-x^2/N}}{\sqrt{\pi N}} \cdot \frac{e^{-y^2/N}}{\sqrt{\pi N}}$$

$$= \frac{e^{-(x^2+y^2)/N}}{\pi N}$$

DEPENDS ONLY ON DIST. FROM ORIGIN.

EXTENSION TO 3D

Can progress from 2D to 3D in much the same way as we got from 1D to 2D.

The probability that any given step is in the $\pm z$ direction is $1/3$, so prob. that M out of N steps are $\updownarrow z$ is $P_M(N;1/3)$, which is sharply peaked around $M = N/3$.

Given M steps $\updownarrow z$, the probability of reaching x, y, z is $P(x,y;N-M)\,P(z;M)$.

Hence

$$P(x,y,z;N) = \sum_{M=0}^{N} P(x,y;N-M)\,P(z;M)\,P_M(N;1/3)$$

$$\simeq P(x,y;N-\tfrac{N}{3})\,P(z;\tfrac{N}{3})\,\underbrace{\sum_M P_M}_{1}$$

$$= P(x,y;\tfrac{2}{3}N)\,P(z;\tfrac{N}{3})$$

$$\simeq P(x;\tfrac{N}{3})\,P(y;\tfrac{N}{3})\,P(z;\tfrac{N}{3})$$

$$= \frac{e^{-(x^2+y^2+z^2)/(2N/3)}}{(2\pi N/3)^{3/2}}$$