PERMUTATIONS & COMBINATIONS

We often need to work out the [a priori] probability of events that could occur in many ways. This involves counting possibilities.

EXAMPLE 1 : 3 coins are flipped. What is the probability of getting 2 heads?

STEP 1 : The size of the sample space [i.e., the number of possible outcomes] $= 2\times2\times2 = 8$.

STEP 2 : # of favourable outcomes [i.e., {HHT, HTH, THH}] $= 3$.

STEP 3 : Assuming that the possible outcomes are all equally likely, $P(2H) = 3/8$.

[ASIDE : Result can be expressed in various ways, $P(2H) = 3/8$
$= 0.375$
$= 37.5\%$
or as odds : "5 to 3 against".]

EXAMPLE 2 : Two dice are rolled. What is probability of getting one or more 1's?

STEP 1 : # of possible outcomes $= 6\times6 = 36$

STEP 2 : favourable outcomes are
$(X,1)$: 5 possibilities with $X \neq 1$
$(1,X)$: 5 " " "
$(1,1)$: 1 possibility.
$\Rightarrow$ # of favourable outcomes $= 11$

STEP 3 : All possibilities equally likely, so $P(\text{at least one } 1) = 11/36$.

GENERALIZATION [of STEP 1]

When we can make n independent choices out of m possibilities,
possible outcomes $= m^n$

E.g., in EXAMPLE 1, $n = 3$ and $m = 2$
$\Rightarrow 2^3$ possible outcomes.

in EXAMPLE 2, $n = 2$ and $m = 6$
$\Rightarrow 6^2$ possible outcomes.

PERMUTATIONS [the order matters]

EXAMPLE 4: How many anagrams can we make from MIKE?

SOLUTION: $4 \times 3 \times 2 \times 1 = 4! = 24$

possibilities for first letter.

possibilities for first of remaining 3 letters.

GENERALIZATION: # of ways to order n distinct objects is $n \times (n-1) \ldots \times 2 \times 1 = n!$

EXAMPLE 5: Anagrams of PEELER.

SOLUTION: Difficulty is the repeated E's. If we temporarily label them E_1, E_2, E_3 we find $6!$ possibilities. But each true anagram corresponds to $3!$ cases differing only in the order of the E's

$\Rightarrow$ # anagrams $= 6!/3!$

EXAMPLE 6: How many [ordered] selections of k objects can be taken from a set of n distinct objects?

SOLUTION: $n(n-1)\ldots(n-k+1) \times \dfrac{(n-k) \times \cdots \times 2 \times 1}{(n-k)!}$

$$= \frac{n!}{(n-k)!} \equiv {}^nP_k$$

COMBINATIONS [order doesn't matter]

How many ways are there to select k items from n different items, if the order doesn't matter?

SOLUTION: If the order mattered, this would be the same as EXAMPLE 6: $\dfrac{n!}{(n-k)!}$

But selections that differ only in the arrangement of the k items are to be counted only <u>once</u>, so total # of unordered selections is

$$\frac{n!}{(n-k)!\,k!} \equiv {}^nC_k \equiv \binom{n}{k}$$

NOTE: We usually describe this as "the number of <u>combinations</u> of n objects taken k at a time".

EXAMPLE 7: How many ways are there of selecting just 2 letters of MIKE?

ANSWER: $\binom{4}{2} = \dfrac{4!}{2!\,2!} = 6.$

PROPERTIES OF $\binom{n}{k}$

Table:

n \ k	0	1	2	3	4
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

Note that:

[1] $\binom{n}{0} = \binom{n}{n} = 1$; $\binom{n}{1} = \binom{n}{n-1} = n$

[2] $\binom{n}{k} = \binom{n}{n-k}$

[3] The PASCAL'S TRIANGLE rule:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

[4] $\binom{n}{k}$ are the coefficients in the binomial expansion of

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Selected derivations:

[1] $\binom{n}{0} = \dfrac{n!}{(n-0)!\,0!} = 1$ (Since $0! = 1$)

$\binom{n}{1} = \dfrac{n!}{(n-1)!\,1!} = n$ (Since $n! = n \times (n-1)!$)

[2] $\binom{n}{n-k} = \dfrac{n!}{(n-(n-k))!\,(n-k)!} = \dfrac{n!}{k!\,(n-k)!} = \binom{n}{k}$

[3] Work from right-hand side:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k+1)!\,(k-1)!}$$

$$= \frac{n!}{(n-k+1)!\,k!}\left\{(n-k+1) + k\right\}$$

$$= \frac{n! \times (n+1)}{(n+1-k)!\,k!}$$

$$= \frac{(n+1)!}{(n+1-k)!\,k!} = \binom{n+1}{k}$$

EXAMPLE 8 : The National Lottery

\# of ways to draw 6 of the 49 numbers

$$= \binom{49}{6} = \frac{49!}{43!\,6!} \simeq 14 \text{ million}$$

\# of jackpot combinations = 1

$\Rightarrow$ P(jackpot win) = 1/14 million.

OTHER PAYOUTS if you get n right and $6-n$ wrong:

$$W_n = \binom{6}{n} \times \binom{43}{6-n}$$

\# of ways of having n of the 6 balls drawn.

\# of ways of having $6-n$ of the 43 that weren't drawn.

$$\Rightarrow P(n \text{ right}) = \binom{6}{n}\binom{43}{6-n} \Big/ \binom{49}{6}$$

EXERCISE (FOR YOU): Evaluate for $n = 3, 4$

[Answers: $P(3) = 0.0177$

$P(4) = 9.69 \times 10^{-4}$]