POISSON DISTRIBUTION

We've looked at problems of "survival" in the presence of constant [and non-constant!] hazards, where a single event [nuclear decay, death, etc.] ends the experiment. But very often we have problems where several events might occur indepently of one another, and the mean rate of occurrence is known:

E.g., Geiger counter clicks in time t due to background radiation.

Number of earthquakes [tsunamis, hurricanes, asteroid impacts...] in a given time.

Number of stars in telescope F.O.V.

Babies born in one week in Stretford.

Number of molecules of gas in a small volume...

All of these might be described by the POISSON DISTRIBUTION.

STRATEGY We already know the probability of no event happening in time t in the presence of a constant hazard rate, α: this is the survival probability

$$P_0(t;\alpha) = e^{-\alpha t}.$$

We'll use it to obtain $P_1(t;\alpha)$, the probability of one event happening in time t, and then extend method to get $P_1, P_2, \ldots P_k(t;\alpha)$.

Note that $\alpha\Delta t$ is both the probability of one event happening in time Δt [i.e., $P_1(\Delta t;\alpha) = \alpha\Delta t$] AND the mean # of events in a short time Δt, because the probability of two events happening in Δt is very small [proportional to $(\Delta t)^2$].

Pedantically,

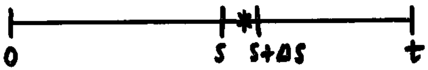
$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k(\Delta t;\alpha)$$
$$\simeq 0 \cdot P_0 + 1 \cdot P_1 = \alpha\Delta t,$$

so α = mean rate of occurrence.

CALCULATION OF P_1

To be definite, suppose the variable is is the background radiation count, so α is the mean # of "clicks" of GM tube per unit time.

Divide the time interval t into 3 parts, assuming that the single count occurs in the short interval $[s, s+\Delta s]$:



This particular scenario has probability

$$\Delta P_1 = P_0(s) \times \alpha\Delta s \times P_0(t-s)$$

prob. of no event up to s

prob. of one event in Δs

prob. of no events in remaining time t−s

$$= e^{-\alpha s} \times \alpha\Delta s \times e^{-\alpha(t-s)}$$

$$= e^{-\alpha t} \times \alpha\Delta s.$$

But ... the event could occur in <u>any</u> small interval Δs, so ...

[CALCULATION OF P_1, CONTINUED]

... the TOTAL probability of one event in time t is

$$P_1(t;\alpha) = \sum \Delta P_1$$

$$= \sum_{\substack{\text{all} \\ \text{intervals}}} e^{-\alpha t}\alpha\Delta s$$

$$= \int_0^t e^{-\alpha t}\alpha\, ds = e^{-\alpha t}\alpha t.$$

[What we've used is the fact that the probabilities of mutually exclusive events can be added: $P(A\cup B) = P(A) + P(B)$ if $A\cap B = \emptyset$. The single event cannot occur in two different intervals Δs (!)]

EXAMPLE 1: Suppose the count rate $\alpha = 2.0\ \text{sec}^{-1}$. Then for $t = 1.0$ sec,

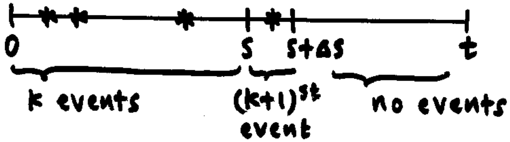
$$P_0 = e^{-\alpha t} = e^{-2} = 0.135$$

$$P_1 = \alpha t e^{-\alpha t} = 2e^{-2} = 0.271$$

$$P_{k>1} = 1 - P_0 - P_1 = 0.594$$

CALCULATION OF P_{k+1} FROM P_k

We can extend the preceding argument. Again, divide the interval into 3 parts:



This particular scenario has probability

$$\Delta P_{k+1} = P_k(s) \times \alpha \Delta s \times P_0(t-s).$$

Add up the contributions for every interval Δs in which the $(k+1)^{st}$ event could happen:

$$P_{k+1}(t;\alpha) = \sum P_k(s) P_0(t-s)\, \alpha\, \Delta s$$

$$\longrightarrow \int_0^t P_k(s) P_0(t-s)\, \alpha\, ds$$

for $\Delta s \to 0$.

E.g.,

$$P_2(t) = \int_0^t P_1(s) P_0(t-s)\, \alpha\, ds$$

$$P_3(t) = \int_0^t P_2(s) P_0(t-s)\, \alpha\, ds, \text{ etc.}$$

[P_{k+1} FROM P_k, CONTINUED]

Now, we already have P_0 and P_1, so

$$P_2(t) = \int_0^t (e^{-\alpha s} \alpha s)(e^{-\alpha(t-s)})\, \alpha\, ds$$

$$= e^{-\alpha t} \int_0^t \alpha^2 s\, ds$$

$$= e^{-\alpha t} \frac{(\alpha t)^2}{2}.$$

Similarly,

$$P_3(t) = \int_0^t \left(e^{-\alpha s} \frac{(\alpha s)^2}{2}\right)\left(e^{-\alpha(t-s)}\right) \alpha\, ds$$

$$= e^{-\alpha t} \int_0^t \frac{\alpha^3 s^2}{2}\, ds$$

$$= e^{-\alpha t} \frac{(\alpha t)^3}{2 \times 3}$$

In the calculation of P_4, the integration of s^3 will give a factor 4 in denominator,

$$P_4(t;\alpha) = e^{-\alpha t} \frac{(\alpha t)^4}{2 \times 3 \times 4}.$$

And in general,

$$\boxed{P_k(t;\alpha) = e^{-\alpha t} \frac{(\alpha t)^k}{k!}}$$

THE POISSON DISTRIBUTION

CHECKS ON P_k

Must make sure that the probabilities sum to unity. Write $\lambda \equiv \alpha t$, then

$$\sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \underbrace{\left\{ \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right\}}_{e^{\lambda}\ \text{[see end]}} = 1 \ \checkmark$$

From the interpretation of α as the mean count rate, the expected # of counts in time t must be $\alpha t = \lambda$:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

term with $k=0$ is zero, so can omit it.

$$= \lambda e^{-\lambda} \underbrace{\left\{ \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right\}}_{e^{\lambda}}$$

[See ADDENDUM]

$$= \lambda \ \checkmark$$

EXERCISE [for You!]: Show that

$\langle k(k-1) \rangle = \lambda^2$, and hence

$\mathrm{var}(k) \equiv \sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2 = \lambda$

[i.e., the mean and the variance of the Poisson distribution are equal]

Solution:

$$\langle k(k-1) \rangle = \sum_{k=0}^{\infty} k(k-1) \cdot \frac{\lambda^k e^{-\lambda}}{k!}$$

terms with $k=0,1$ are zero

$$= \underbrace{\left(\sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} \right)}_{e^{\lambda}} \lambda^2 e^{-\lambda} = \lambda^2 .$$

But $\underbrace{\langle k(k-1) \rangle}_{\lambda^2} = \langle k^2 \rangle - \underbrace{\langle k \rangle}_{\lambda}$

$$\Rightarrow \langle k^2 \rangle = \lambda^2 + \lambda .$$

Hence

$$\mathrm{var}(k) = \langle k^2 \rangle - \langle k \rangle^2$$

$$= (\lambda^2 + \lambda) - \lambda^2 = \lambda .$$

ADDENDUM:

We want to show that $Y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

is simply e^x.

First differentiate Y with respect to x:

$$\frac{dY}{dx} = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{k\,x^{k-1}}{k!}$$

(the term with $k=0$ is zero, so we omit it in the next line)

$$= \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$$

$$= \sum_{\ell=0}^{\infty} \frac{x^\ell}{\ell!}$$ (with new summation variable $\ell = k-1$)

i.e., $\frac{dY}{dx} = Y$. (A)

Also note that $Y(0) = 1$. Now solve (A):

$$\int \frac{1}{Y}\, dY = \int dx$$

$$\Rightarrow \ln Y = x + C,$$

but $C = 0$, because $Y=1$ when $x=0$.

Hence $Y = e^x$.