HAZARD RATE

For the problem of RA decay of a nucleus we found a survival probability that was exponential in time: $P(t) = e^{-t/T}$

and found that the p.d.f. for the decay time was exponentially distributed:

$$f(t) = -\frac{dP}{dt} = \frac{1}{T} e^{-t/T}. \qquad \text{[LECTURE 8]}$$

[Similarly for molecular collisions, $P(s) = e^{-s/\lambda}$ and the p.d.f. for the distance travelled to the first collision would be $f(s) = -\frac{dP}{ds} = \frac{1}{\lambda} e^{-s/\lambda}$.]

The probability of decay in a time interval $(t, t+\Delta t)$ is $f(t)\Delta t$, which can be interpreted as

$$f(t)\Delta t = e^{-t/T}\frac{\Delta t}{T} = \underbrace{P(t)}_{\text{prob. of survival to time } t}\ \underbrace{\frac{\Delta t}{T}}_{\text{prob. of decay in subsequent instant } \Delta t} \qquad \text{(A)}$$

We can think of $\frac{1}{T}$ as the "probability per unit time" for the decay process [given survival to t], or as a HAZARD RATE for the process.

[HAZARD RATE, CONTINUED]

More generally, we would need to consider problems where the hazard rate was <u>not</u> constant: e.g., the probability per unit time of a light bulb failing [or a person dying] increases with its age; a soldier's risk of death or injury will be greater in wartime; etc.

How does the survival probability depend on the hazard rate, $\alpha(t)$?

We can simply adapt (A) to this new problem:

$$\underbrace{f(t)\Delta t}_{\text{prob. of failure in interval } (t, t+\Delta t)} = \underbrace{P(t)}_{\text{prob. of survival to time } t} \times \underbrace{\alpha(t)\Delta t}_{\text{prob. of failure in subsequent instant } \Delta t} \qquad \text{(B)}$$

Since $f(t) = -\frac{dP}{dt}$, (B) gives a differential equation for P:

$$\boxed{\frac{dP}{dt} = -f(t) = -\alpha(t)P(t)} \qquad \text{(C)}$$

[HAZARD RATE, CONT$^{\underline{D}}$]

SOLUTION OF ©: Proceed as in LECTURE 8, where the hazard rate [for molecular collision, per unit distance] was $1/\lambda$ —

$$\frac{dP}{dt} = -\alpha(t)\,P$$

$$\Rightarrow \quad \frac{1}{P}\frac{dP}{dt} = -\alpha(t)$$

$$\Rightarrow \quad \int_{P(0)}^{P(t)} \frac{dP'}{P'} = -\int_0^t \alpha(t')\,dt'$$

1 : initial condition

$$\Rightarrow \quad \left[\ln P'\right]_{P(0)}^{P(t)} = \ln\left(P(t)/P(0)\right) = -\int_0^t \alpha(t')\,dt'$$

$$\Rightarrow \quad \boxed{P(t) = e^{-\int_0^t \alpha(t')\,dt'}}$$

CHECKS: For $\alpha = 1/T$ [constant] this reproduces $P(t) = e^{-t/T}$ for RA decay. ✓

Should also check by explicitly calculating

$$\frac{dP}{dt} = \underbrace{e^{-\int_0^t \alpha(t')\,dt'}}_{=\;P(t)} \times \underbrace{\frac{d}{dt}\left\{-\int_0^t \alpha(t')\,dt'\right\}}_{\times\,(-\alpha(t))} \quad ✓$$

[HAZARD RATE, CONTINUED]

EXAMPLE 1 : Suppose the hazard rate for failure of a lightbulb is $\alpha(t) = \epsilon t^2$, where $\epsilon = 2.5\times10^{-4}\,[\text{month}]^{-3}$, and t is measured in months. Calculate the probability that the bulb works for at least 24 months.

SOLUTION : We need the SURVIVAL PROBABILITY $P(24)$. Here

$$P(t) = e^{-\int_0^t \epsilon t'^2\,dt'} = e^{-\epsilon t^3/3}$$

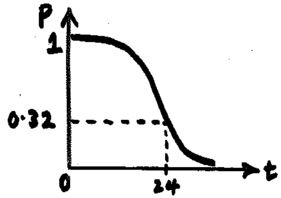
So

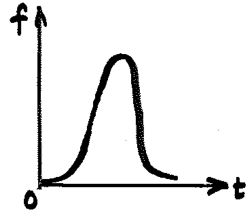
$$P(24) = \exp\left[-2.5\times10^{-4}\times 24^3/3\right] = e^{-1.152} \simeq \underline{\underline{0.32}} \quad (32\%)$$

EXAMPLE 2 : Find the p.d.f. $f(t)$ for lightbulb failure.

SOLUTION : use either $f = -\frac{dP}{dt}$, or

$$f = \alpha P = \epsilon t^2 e^{-\epsilon t^3/3}$$





[HAZARD RATE, CONT^{D}]

MOST LIKELY TIME OF FAILURE: i.e., the time $\hat{t}$ at which f has its greatest value.

A general approach is to use

$f = \alpha P$ and $\frac{dP}{dt} = -f$, so that

$$\frac{df}{dt} = \frac{d\alpha}{dt}P + \alpha\frac{dP}{dt} \quad \text{[derivative of } \alpha P\text{]}$$

$$= \frac{d\alpha}{dt}P + \alpha\{-\alpha P\}$$

$$= \left\{\frac{d\alpha}{dt} - \alpha^2\right\}P = 0 \quad \text{at } t = \hat{t}.$$

Hence $$\boxed{\frac{d\alpha}{dt} = \alpha^2 \quad \text{at } t = \hat{t}.}$$

In general this equation for $\hat{t}$ may have several solutions: you have to check which one corresponds to maximum f.

EXAMPLE 3: Find the most likely time of failure for the lightbulb of EXX 1 & 2.

SOLUTION: see opposite.

SOLUTION:

$$\alpha(t) = \epsilon t^2 \Rightarrow \frac{d\alpha}{dt} = 2\epsilon t$$

Now, $\frac{d\alpha}{dt} = \alpha^2$ at $t = \hat{t}$, so

$$2\epsilon\hat{t} = (\epsilon\hat{t}^2)^2 = \epsilon^2\hat{t}^4$$

$$\Rightarrow \hat{t}\left(\frac{2}{\epsilon} - \hat{t}^3\right) = 0$$

$$\Rightarrow \hat{t} = 0 \text{ or } \left(\frac{2}{\epsilon}\right)^{1/3}.$$

We can rule out $\hat{t} = 0$ as the solution, as $f(0) = \alpha(0)P(0) = 0$, which can't be the maximum.

Hence $$\hat{t} = \left(\frac{2}{\epsilon}\right)^{1/3} = \left(\frac{2}{2.5 \times 10^{-3}}\right)^{1/3}$$

$$= 8000^{1/3} = \underline{20 \text{ months}}.$$