10471: RANDOM PROCESSES

Lecturer : Mike Godfrey

Location : Room 7.17, Schuster Lab

E-mail : m.godfrey@manchester.ac.uk

WHAT DO WE MEAN BY 'RANDOM'?

E.g.: not predictable (so no "pattern" that would let us predict the results);
uncontrolled? (we don't "fix" the conditions to get the result we want);

Examples of 'random' events/processes:

Electronic transitions in atoms, Radioactive decay } (intrinsically random, in quantum mechanics)

Rolling die, flipping coin, etc. } (we don't know the initial conditions)

Chaotic systems (these rapidly "forget" their initial conditions — in the sense that knowing them to a fixed level of precision doesn't allow long-term prediction)

Etc.

Examples of 'non-random' events/processes:

Motion of planets (predictable over thousands of years)

An encrypted message? (it might look like a random sequence of symbols — but not if you have the key!)

DEFINING PROBABILITY

There are various schools of thought on what probability "means", but there is agreement on its properties, which we abstract from the ...

EMPIRICAL VIEWPOINT:

PROBABILITY = RELATIVE FREQUENCY

We do an experiment [roll a die?] whose possible outcomes are $\{x_1, x_2, \dots x_M\} = \mathcal{E}$
$\equiv$ "sample space"
$\equiv$ "universal set"

Repeat N times $[N \gg 1]$: x_i occurs N_i times.

DEFINE the probability P_i [or $P(x_i)$] by

$$\boxed{P_i \;[= P(x_i)] = \lim_{N\to\infty} N_i/N}$$

DEDUCTIONS: $0 \leq P(x_i) \leq 1$

0: x_i never occurs; 1: x_i always occurs

Probability of getting any result in subset A of $\mathcal{E}$ is

$$P(A) = \underbrace{\Big(\sum_{x_i\in A} N_i\Big)}_{\text{total number of times we got a result in } A.}/N = \sum_{x_i\in A} P(x_i)$$

[EMPIRAL VIEWPOINT/DEDUCTIONS ...]

If $\bar{A}$ is the complement of A in $\mathcal{E}$ [i.e., $x \in \bar{A} \iff x \notin A$],

$$P(A) + P(\bar{A}) = 1.$$

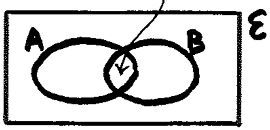
More generally, if A and B are <u>mutually exclusive</u> [i.e. $A \cap B = \emptyset$, the empty set],

$$P(A \cup B) = P(A) + P(B).$$

Still more generally, if A and B are <u>not</u> mutually exclusive

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Venn diagram:



Last term "subtracts out" events that would be counted twice in $P(A) + P(B)$.

[For basic set theory <u>notation</u>, see Jordan & Smith "Mathematical Techniques" 3rd ed., Chapter 35. We won't use the <u>algebra</u> of sets explicitly.]

<u>AXIOMATIC APPROACH</u>

Probability is a real, non-negative function of a <u>set</u>, with properties

$$P(\mathcal{E}) = 1$$

$$P(A \cup B) = P(A) + P(B) \quad \text{for disjoint sets } A \text{ and } B \; [\text{i.e. } A \cap B = \emptyset]$$

Axiomatic approach doesn't tell us <u>how</u> to assign probabilities to sets, but it summarizes the rules for manipulating them.*

It overcomes an obvious problem of the empirical approach: no experiment is needed, and we don't need to assume the existence of the [impractical!] limit $N \to \infty$...

On the other hand, it gives no indication of what probabilities might be useful for [e.g., predicting results of future experiments]

*[Warning: some books give <u>more</u> axioms, but they are all reducible to the two given here!]

A PRIORI PROBABILITIES

Assignment of probabilities without doing an experiment.

EXAMPLE: A cubical die

The cube is symmetrical in its important physical properties, such as shape and distribution of mass. [The spots are lightweight dabs of paint!] "No reason" to expect one face to come up more frequently than another, so assign $P(1) = P(2) = \cdots P(6) = 1/6$.

MORE GENERALLY: If an experiment has K different outcomes, all of which are physically similar, assign prob. $1/K$ to each.

Note that we should always try to argue from our knowledge of a system [e.g. its symmetry], not from ignorance.

EXAMPLE: What's the chance of rain in Manchester on 17 Aug 2010?

BAD ANSWER: 50% [there are two possibilities, and no reason to choose between them...] X

A more sensible approach would be to look at meteorological records for mid August and use the relative frequency definition — assuming you have data for many years past!