PHYS40352 Solid State Physics Examples 3

1. *Optional problem:* In lectures we found an expression for the Pauli spin susceptibility of an electron gas at low density and high temperature. For applications to simple metals at ordinary temperatures $T \ll E_F/k_B$, it is also useful to consider the case of a free-electron gas at zero temperature in a magnetic field B.

   If the up- and down-spin electrons are at equilibrium at $T = 0$, the total energy of the electron gas must be a minimum, and the energy will not be reduced by simply reversing the spin of one electron. This means that the *maximum* energy of a spin-up electron must equal that of a spin-down electron,

   $$E_{F\uparrow} + \mu_B B = E_{F\downarrow} - \mu_B B,$$

   where B is in the positive z-direction and the signs take account of the fact that the electron spin is in the opposite direction to its magnetic moment; $E_{F\uparrow}$ and $E_{F\downarrow}$ are the usual Fermi kinetic energies given, e.g., by

   $$E_{F\uparrow} = \hbar^2 k_{F\uparrow}^2/2m, \quad \text{where} \quad k_{F\uparrow} = \left(6\pi^2 N_\uparrow/V\right)^{1/3}.$$

   Assume that $N_\uparrow$ and $N_\downarrow$, the numbers of up- and down-spin electrons, are close to their zero-field values $\frac{1}{2}N$, and expand $E_{F\uparrow}$ and $E_{F\downarrow}$ to first order in the small quantities $(N_\uparrow - \frac{1}{2}N)$ and $(N_\downarrow - \frac{1}{2}N)$. Hence show that the spin susceptibility *per electron* is

   $$\chi_{\text{spin,el}} = -\mu_0\mu_B(N_\uparrow - N_\downarrow)/NB = 3\mu_0\mu_B^2/2E_F\,,$$

   where E_F is the Fermi energy in zero field.

2. Use the expression given at the end of Q.1 to calculate the Pauli spin susceptibility *per unit volume* of sodium,

   $$\chi_{\text{spin}} = n\,\chi_{\text{spin,el}}\,,$$

   where $n = 2.65 \times 10^{28}\,\text{m}^{-3}$ is the number density of electrons in sodium.

   The experimental result for χ_{spin} is approximately 1.4×10^{-5} for sodium; your calculated result should be of the same order of magnitude. The discrepancy with experiment is significant and is at least partly due to the neglected effects of the electron–electron interaction.

3. Apply Hund's rules to find J, L, S for the ground-states of the rare earth ions Ce^{3+} (configuration $4f^1$) and Er^{3+} (configuration $4f^{11}$). Hence verify two of the results for

   $$p_{\text{th}} = g(J, L, S)\sqrt{J(J+1)}$$

   given in Table 3.1 of the online notes. [Use the formula for the Landé g-factor quoted elsewhere in the notes.]

4. By using a procedure similar to the one used in Section 3.3.1 of your notes, show that, in a magnetic field B, the thermal average M of the magnetic moment of an ion with $J = 1$ is given by

   $$M = \left\{\frac{e^x - e^{-x}}{e^x + 1 + e^{-x}}\right\} g\mu_B\,,$$

   where $x \equiv g\mu_B B/k_B T$. Plot (or carefully sketch) the function of x given in curly brackets, and note the general similarity to the tanh function.

5. Suppose that the interactions of three electrons in a triangular molecule can be described by the Heisenberg Hamiltonian

   $$\hat{H} = J(\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 + \hat{\boldsymbol{s}}_2 \cdot \hat{\boldsymbol{s}}_3 + \hat{\boldsymbol{s}}_3 \cdot \hat{\boldsymbol{s}}_1)/\hbar^2.$$

   Express $\hat{H}$ in terms of $(\hat{\boldsymbol{s}}_1 + \hat{\boldsymbol{s}}_2 + \hat{\boldsymbol{s}}_3)^2$ and (by considering the rules for the addition of angular momentum) show that the ground state energy is $\frac{3}{4}J$ in the ferromagnetic case $J < 0$. What is the degeneracy of the ground state? Write down one of the ground state wave functions, e.g. the one that corresponds classically to all the spins being aligned.

   The case $J > 0$ is usually described as being *frustrated*, because on a triangle you cannot arrange for all pairs of adjacent spins to be antiparallel. Show that the ground state energy is $-\frac{3}{4}J$, and interpret the wave function in this case. [Note that a pair of electrons in a singlet state could be interpreted as a *valence bond*: the ground state of the hydrogen molecule is a spin singlet.]

6. In the mean-field model of a ferromagnet (with spins $s = \frac{1}{2}$), we found

   $$m = \tanh\left(mT_c/T\right), \qquad (1)$$

   where T_c is the critical temperature and m is the magnetization expressed as a fraction of its maximum possible value, so that $|m| \leq 1$. How would (1) be changed by the presence of an applied magnetic field B? Hence show that, for $T = T_c$ and in a very weak field B, m will be proportional to $B^{1/3}$.

   [*Note:* m is small, so you can use the approximation $\operatorname{arctanh} m \simeq m + \frac{1}{3}m^3$. In experiments, the observed behaviour of m is closer to $B^{1/5}$: mean-field theory gives only approximate values for the critical exponents.]