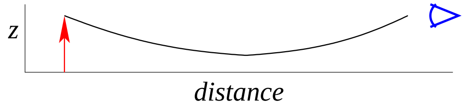
PHYS 30672 MATHS METHODS Examples 4

1. Mirages occur where the refractive index of the air $n(z)$ increases with height z above the ground. According to Fermat's principle, the path of light travelling from A to B is the one minimizing the travel time between these points.



(i) Show that for this path,

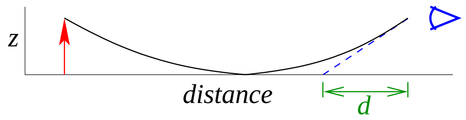
$$\frac{dz}{dx} = \sqrt{A^2 n(z)^2 - 1}\,.$$

(ii) If $n(z) = n_0(1 + \alpha z)$, show that

$$An(z) = \cosh\left(An_0\alpha[x - x_0]\right)$$

where x_0 is a constant.
(iii) For a ray just grazing the surface at $x = 0$ show that

$$1 + \alpha z = \cosh(\alpha x).$$



(iv) Assuming αx is small, show that for an observation point P at height z, the grazing ray appears to come from a point at distance $d = (z/2\alpha)^{1/2}$.

2. Water (and fish, which may be neglected) are contained in a large glass vessel with vertical sides. Owing to surface tension, the water rises a little near the sides of vessel, forming a meniscus. The problem is to find the height z of the water surface as a function of distance x from the side of the vessel.

The shape of the meniscus $z(x)$ is determined by minimizing the static energy, which is equal to the surface energy, $\sigma \times$ [area], plus the gravitational potential energy. The origin, $z = 0$, can be chosen to be the water level far from the side of the vessel; any change to this level will be negligible if the vessel is sufficiently large. Use a first integral of the Euler–Lagrange equations to show that

$$\tfrac{1}{2}\rho g z^2 + \frac{\sigma}{\sqrt{1+p^2}} = \text{const.},$$

where $p = dz/dx$ and ρ is the density of water.

Assuming that the contact angle between water and glass is zero, show that the water creeps a distance $\sqrt{2\sigma/\rho g}$ up the glass.

If you have plenty of energy and time, obtain a *parametric* solution for the shape of the meniscus; e.g., in terms of the angle θ that the water surface makes with the horizontal.

3. Using polar coordinates (R, θ, ϕ) on the surface of a sphere of radius R, show that the shortest path $\theta(\phi)$ joining two points on the surface satisfies the equation

$$\left(\frac{d\theta}{d\phi}\right)^2 = A\sin^4\theta - \sin^2\theta\,,$$

where A is constant.

Aside: If you want to solve this, try rewriting it as a differential equation for $\cot\theta$. There is an analytic solution, which of course corresponds to a *great circle* path round the sphere.

4. In relativistic quantum mechanics, the Lagrangian density for a neutral π-meson is given by

$$\Lambda = \tfrac{1}{2}\left(\dot\phi^2 - \nabla\phi\cdot\nabla\phi - \mu^2\phi^2\right)\,,$$

where μ is the pion mass, $\phi(\boldsymbol{r}, t)$ is a real wavefunction, and units have been chosen such that $c = 1$. Assuming Hamilton's principle $\delta S = 0$, where the action

$$S[\phi] = \iint \Lambda\, d^3\boldsymbol{r}\, dt\,,$$

find the wave equation satisfied by ϕ.

5. A solid spherical planet of radius R rotates with angular velocity ω, and is covered with a layer of water of depth $h(\theta) \ll R$, where θ is the polar angle.

For a small volume of water dV in the layer, at position $y(\theta)$ above the solid surface, $0 < y(\theta) < h(\theta)$, the gravitational potential energy is $g\rho\, y\, dV$ and the rotational potential energy is $-\rho\omega^2R^2\sin^2\theta\, dV/2$, where ρ is the density of water, g is the acceleration due to gravity and the approximation $y \ll R$ has been used. In the same approximation, the volume dV is $dV = R^2\sin\theta\, dy\, d\theta\, d\phi$.

Show that the total potential energy of the water is

$$E = \pi\rho R^2 \int_0^\pi \left(gh^2 - \omega^2R^2h\sin^2\theta\right)\sin\theta\, d\theta\,.$$

Minimize E subject to the condition of constant volume V and hence derive an expression for the depth of water $h(\theta)$.

This was the second part of an exam question set in June 1999.

6. The Hamiltonian operator for a particle in the region $0 < x < \infty$ is

$$\hat{H} = -\frac{d^2}{dx^2} + x\,,$$

where units have been chosen such that $\hbar^2 = 2m$. The boundary conditions for the wave function are

$$\psi \to 0 \text{ as } x \to \infty \text{ and } \frac{d\psi}{dx} = 0 \text{ at } x = 0\,.$$

Choosing the trial function $\psi = (\beta x + 1)\exp(-\alpha x)$, where α and β are real constants, determine a value for β such that the boundary conditions are satisfied. Show that for this (real) trial function,

$$\int_0^\infty \psi \hat{H} \psi\, dx = \frac{\alpha}{4} + \frac{9}{8\alpha^2}\,.$$

Use the Rayleigh–Ritz method to estimate a value for the ground state energy.

You may assume the integral

$$\int_0^\infty x^n \exp(-sx)\, dx = \frac{n!}{s^{n+1}}.$$

Aside: You might have been troubled by the unusual boundary condition imposed at $x = 0$. If so, you can imagine that the potential is $|x|$ and that we're looking for the ground state wave function, which is an even function of x and hence has zero derivative at $x = 0$. We then restrict *our attention*—rather than the particle—to the region $x > 0$.