PHYS 30672 MATHS METHODS Examples 3

Note: 2, 4, 5(a), 10 and 11 could be attempted over the vacation. 1 and 3(b) require conversion to a differential equation, which is not covered in the first two lectures on integral equations; 7 can also be tackled this way, or by careful use of an integral transform — note that it is *not* a standard case of a displacement kernel. 3(a) and 8 ask specifically for a Laplace transform solution. 6 and 9 do not require any additional knowledge, but they may be more challenging.

1. Solve the Volterra equation

$$f(x) = x^2 + 1 + \int_0^x y f(y)\, dy$$

by converting it to a first-order ODE. You will have to deduce the initial value $f(0)$ from the integral equation.

Ans: $f(x) = 3\, e^{x^2/2} - 2$

2. Solve the Fredholm equation

$$f(x) = x + \int_{-\pi}^{\pi} \cos(x-y)\, f(y)\, dy$$

by any method that seems appropriate.

Ans: $f(x) = x - \dfrac{2\pi}{\pi - 1} \sin x$

3. (a) Solve the Volterra equation

$$f(x) = x + \int_0^x (x-y) f(y)\, dy$$

by using the Laplace transform method and the convolution theorem.

Ans: $f(x) = \sinh x$

(b) Solve the equation again by converting it to a differential equation, deducing the initial values $f(0)$ and $f'(0)$ from the integral equation and its first derivative.

4. Find the eigenvalues and eigenfunctions of the homogeneous Fredholm equation

$$f(x) = \lambda \int_0^1 (x+y) y\, f(y)\, dy\,.$$

Hints: Use the separable kernel method, or make an educated guess at the form of the solution. If you don't trust yourself not to make an algebraic slip, try using Mathematica: define a suitable `f[x]` containing arbitrary constants and use the function `SolveAlways[f[x]==<RHS>,{x}]`

Ans: $\lambda = -24 \pm 18\sqrt{2}$, $f(x) = \sqrt{2}\, x \pm 1$ (unnormalized)

5. Solve the Fredholm equation

$$f(x) = x + \lambda \int_0^1 (x+y) y\, f(y)\, dy$$

by using (a) the separable kernel method and (b) the Hilbert–Schmidt eigenfunction expansion of the resolvent kernel. The parameter λ is not equal to either of the eigenvalues found in Q.4.

Hint for (b): The equation satisfied by $\sqrt{x} f(x)$ has a symmetric kernel, $K(x,y) = \sqrt{x}(x+y)\sqrt{y}$; modify your results from Q.4 appropriately.

6. Consider the integral equation

$$f(x) = e^{-|x|} + \lambda \int_0^\infty f(y) \cos(xy)\, dy\,.$$

What is the symmetry of the solution? Find $f(x)$ by taking the Fourier transform of each side, making use of the symmetry.

7. Solve the integral equation

$$f(x) = e^{-|x|} + \lambda \int_0^\infty e^{-|x-y|} f(y)\, dy \quad \text{where } -\infty < x < \infty \text{ and } \lambda > \tfrac{1}{2}.$$

Hints: Treat the cases $x > 0$ and $x < 0$ separately. The case $x < 0$ can be solved by inspection. For $x > 0$, differentiate the integral equation twice; it is helpful to split the range of integration first.

8. In lectures, we showed that normal projection of a spherically-symmetric density function $\rho(r)$ on to the (x, y) plane leads to a 2D density function

$$\sigma(R) = 2 \int_R^\infty \frac{r\, \rho(r)}{\sqrt{r^2 - R^2}}\, dr\,.$$

Use the substitutions $x = 1/R^2$ and $y = 1/r^2$ to rewrite this in the form

$$g(x) = \int_0^x \frac{f(y)}{\sqrt{x-y}}\, dy\,,$$

which is known as *Abel's integral equation.* Take the Laplace transform of each side to show that the solution is

$$f(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{g(y)}{\sqrt{x-y}}\, dy\,.$$

Note: You won't need the Bromwich integral to solve this problem, but it is helpful to note that

$$\int_0^\infty x^{-1/2} e^{-sx} dx = (\pi/s)^{1/2}\,,$$

which you could prove by making the change of variable $x = u^2$.

9. The integral equation

$$\nu(x) = \tfrac{1}{2}\, e^{-x} + \tfrac{1}{2} \int_0^L e^{-|x-y|}\, \nu(y)\, dy \qquad \text{for} \qquad 0 \le x \le L$$

is a version of the integral equation for molecular flow down a cylindrical tube of length L, in which the transfer probability $p(|x-y|)$ has been approximated by the exponential function. Prove that the solution is a linear function of x.

For a harder problem, show that the solution of

$$\nu(x) = s(x) + \int_0^L p(|x-y|)\, \nu(y)\, dy\,, \quad \text{where} \quad s(x) = \int_x^\infty p(y)\, dy\,,$$

is a linear function of x **only** if p is the exponential function. [You should assume that terms of order $p(L)$ are negligible compared with $p(x)$ for $L \gg x$.] Hence, despite appearances, the numerical solution obtained in `MolecularFlow.nb` cannot be exactly linear.

10. Schrödinger's equation for a particle bound by an attractive 1D potential $V(x) < 0$ can be written in the form

$$-u''(x) + \lambda W(x) u(x) = -\gamma^2 u(x)\,,$$

where $\lambda W \equiv 2mV(x)/\hbar^2 \to 0$ for $x \to \pm\infty$. The constant $\lambda > 0$ is a measure of the strength of the potential. By finding the Green's function satisfying

$$\frac{\partial^2}{\partial x^2} G(x,y) - \gamma^2 G(x,y) = \delta(x-y)$$

with appropriate boundary conditions for $x \to \pm\infty$, show that

$$u(x) = \lambda \int_{-\infty}^{\infty} G(x,y)\, W(y)\, u(y)\, dy\,,$$

where $G(x,y)$ is a function only of $x-y$. This eigenvalue equation for $u(x)$ can be used to find the strengths of potential for which the original Schrödinger equation has a bound state with energy $-\hbar^2\gamma^2/2m$.

(a) By considering $u(0)$, show that the potential $\lambda W = -\lambda\delta(x)$ has a bound state for any $\lambda > 0$ and that $\gamma = \lambda/2$ for this state.

(b) For the potential $\lambda W = -\lambda\delta(x) - \lambda\delta(x-a)$, write down the linear equations satisfied by $u(0)$ and $u(a)$. From these equations, show that there is always at least one bound state if $\lambda > 0$ and that there will be a second bound state if $\lambda > 2/a$. Sketch the two bound-state wave functions.

11. Consider the integral equation

$$f(x) = g(x) + \lambda \int_{-\infty}^{\infty} e^{-|x-y|} f(y)\, dy \quad \text{for} \quad \lambda < \tfrac{1}{2}\,.$$

Assuming that the Fourier transforms of f and g exist, show that they are related by

$$\tilde{f}(k) = \frac{k^2+1}{k^2+1-2\lambda}\, \tilde{g}(k)\,.$$

Hence show that the solution $f(x)$ is

$$f(x) = g(x) + \frac{\lambda}{\gamma} \int_{-\infty}^{\infty} e^{-\gamma|x-y|} g(y)\, dy\,, \quad \text{where } \gamma = \sqrt{1-2\lambda}\,.$$

Does this solution make sense to you for $|\lambda| \ll 1$?