PHYS 30672 MATHS METHODS Examples 1

1. Show that the following limit could be used as a definition of the Dirac δ function:

$$\lim_{a\searrow 0} \Delta_a(x)\,, \quad \text{where} \quad \Delta_a(x) = \frac{1}{\sqrt{\pi}a}\exp(-x^2/a^2).$$

Hint: Assume that $\Delta_a(x)$ is multiplied by a function $f(x)$ that can be expanded in a Taylor series about $x = 0$, and consider the integral $\int f(x)\Delta_a(x)\,dx$ in the limit $a \searrow 0$. More details are worked out in the Mathematica notebook dirac.nb on the web site, but you don't need these to obtain the final result.

2. By using the Fourier representation of a δ function, show that *Parseval's theorem* holds:

$$\int_{-\infty}^{\infty} f^*(x)\,g(x)\,dx = \int_{-\infty}^{\infty} \tilde{f}^*(k)\,\tilde{g}(k)\,\frac{dk}{2\pi}\,,$$

where

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx \quad \text{and} \quad f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\,e^{ikx}\,\frac{dk}{2\pi}\,.$$

3. Show that the integration operators L_1 and L_2 defined by

$$(L_1 f)(x) = \int_0^x f(x')\,dx',$$

$$(L_2 f)(x) = \int_0^1 G(x,x')f(x')\,dx',$$

are linear operators.

4. If one were to define the adjoint of a linear operator L by

$$(\boldsymbol{e}_i\,,\,L^\dagger\,\boldsymbol{u}_k) = (\boldsymbol{u}_k\,,\,L\,\boldsymbol{e}_i)^* = (L\,\boldsymbol{e}_i\,,\,\boldsymbol{u}_k)$$

where $\boldsymbol{e}_i\,,\,\boldsymbol{u}_k$ are basis vectors in the domain and codomain respectively, show that this implies

$$(\boldsymbol{a}\,,\,L^\dagger\,\boldsymbol{b}) = (\boldsymbol{b}\,,\,L\,\boldsymbol{a})^* = (L\,\boldsymbol{a}\,,\,\boldsymbol{b})$$

where $\boldsymbol{a}\,,\,\boldsymbol{b}$ are arbitrary vectors in the domain and codomain, respectively.

5. Consider the generalized eigenvalue problem

$$L\,y(x) = \lambda\,\rho(x)\,y(x)$$

where L is an Hermitian differential operator, $\rho(x)$ is a real, positive definite weight function. Prove that the eigenvalues are real, and that eigenfunctions $u(x)$, $v(x)$ corresponding to different eigenvalues satisfy the orthogonality condition

$$\int dx\,\rho(x)\,u(x)^*\,v(x) = 0\ .$$

6. Chebyshev polynomials $T_n(x)$ of type 1 are polynomials of order n which satisfy the differential equation

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0\,. \qquad (1)$$

in the range $-1 < x < 1$. Multiply this equation by $(1-x^2)^{-1/2}$ and show that the resulting equation is of Sturm–Liouville form. What is the form of the orthogonality relation for these functions?

For $n = 0$, equation (1) has the trivial solution $T_0(x) = 1$. Find a second solution in the form of an indefinite integral, and then evaluate the integral to obtain its explicit form.

7. Show that the eigenvalues λ of the Sturm-Liouville equation

$$(-py')' + qy = \lambda\rho y$$

are all positive if $p(x)$, $q(x)$ and $\rho(x)$ are all positive for $a \le x \le b$ and $y(a) = y(b) = 0$. (Assume that $y(x)$ is real.)

Hint: Start by multiplying the equation by y and integrate from a to b using integration by parts.

8. The function $u(x)$ is a solution of the Sturm–Liouville equation

$$-\frac{d}{dx}(p(x)\,u') + qu = \lambda\rho u, \quad \text{with } p > 0 \text{ and } \rho > 0,$$

satisfying the boundary condition $u'/u = c$ at $x = a$. We also know that $u(x)$ has a zero at $x = X$.

If λ changes by a small amount to μ, a new solution $v(x)$ "close" to $u(x)$ is obtained, satisfying the same boundary condition at $x = a$, with a corresponding zero at $x = Y$. [continued]

Show that if $\mu > \lambda$ then $X > Y$.

Hint: Use the basic relation $\int_a^X dx\,(vLu - uLv) = [p(uv' - vu')]_a^X$ to show $u'v > 0$ at $x = X$, and consider its implications for the zeroes by sketching the behaviour of $u(x)$ and $v(x)$ close to $x = X$ for the cases $u'(X) > 0$, $u'(X) < 0$.

Alternative approach: Regard u as a function of two variables, x and λ. Differentiate the S-L equation with respect to λ to find the inhomogeneous equation satisfied by $\partial u/\partial\lambda$. Then use the same "basic relation" as above, with v replaced by $\partial u/\partial\lambda$. Note that X is now a function of λ, defined implicitly via $u(X(\lambda),\lambda) = 0$, and that you are aiming to show that $dX/d\lambda < 0$.