ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Mathematical Methods for Physics

xx xxx 2015, 0.00 a.m. - 1.30 a.m.

Answer **ANY TWO** questions

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. A linear differential operator L_x acts on functions $u(x)$ defined for $x \in [a, b]$.

(a) The operator L_x has the Sturm–Liouville form

$$L_x u \equiv -\frac{d}{dx}\left(p\,\frac{du}{dx}\right) + qu\,,$$

where p and q are real functions of x.
Show that for any two functions u and v,

$$vL_x u - uL_x v = (puv' - pvu')'\,.$$

Hence derive a general condition that must be satisfied for L_x to be Hermitian.

[5 marks]

(b) Consider the Sturm–Liouville operator defined by

$$L_x u \equiv xu'' + u' - \frac{4}{x}u\,,$$

where $u(0)$ and $u'(0)$ are finite and $u(1) = 0$. Show that these conditions are sufficient to ensure that L_x is Hermitian.

[3 marks]

(c) Write down the differential equation satisfied by the Green's function $G(x, y)$ for a linear operator L_x and derive the Green's function solution of the inhomogeneous equation $L_x u = f(x)$.

[5 marks]

(d) Find the Green's function for the operator L_x and the boundary conditions defined in part (b). As part of your working, you should consider solutions of the homogeneous equation $L_x u = 0$ that have the power-law form $u = x^k$.

[8 marks]

(e) Use the Green's function found in part (d) to solve the inhomogeneous equation

$$L_x u = x^2\,.$$

[4 marks]

2. (a) In the eigenvalue problem defined by

$$u_n(x) = \lambda_n \int_0^1 K(x,y)\, u_n(y)\, dy\,,$$

the kernel $K(x,y)$ is a real, symmetric function of its arguments. By using an appropriate definition of the scalar product of two functions, show that the eigenvalues λ_n are real and that eigenfunctions corresponding to different eigenvalues are orthogonal.

[5 marks]

(b) Derive a formula for $K(x,y)$ in terms of the eigenfunctions of the integral equation given in part (a).

[5 marks]

(c) Consider the eigenvalue problem

$$u_n(x) = \lambda_n \int_0^{1-x} u_n(y)\, dy$$

for $0 \le x \le 1$. Show that this can be written as an eigenvalue problem of the same form as in part (a), with a symmetric kernel $K(x,y)$.

[3 marks]

(d) For the eigenvalue problem defined in part (c), find a differential equation satisfied by the eigenfunctions and derive the boundary conditions for the functions $u_n(x)$.

[5 marks]

(e) Solve the differential equation derived in part (d) and hence derive the eigenvalue spectrum

$$\lambda_n = \left(2n + \tfrac{1}{2}\right)\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots.$$

[5 marks]

(f) Use your solutions from part (e) to verify explicitly that the eigenfunctions u_n and u_m are orthogonal for general $m \neq n$.

[2 marks]

3. (a) In a variational problem $\delta U[y] = 0$, the functional U has the form

$$U[y] = \int_a^b f(y, y_x, y_{xx}, x)\, dx\,,$$

where $y_x = dy/dx$ and $y_{xx} = d^2y/dx^2$. The variations $y(x)$ are unconstrained for $a < x \le b$, but $y(a)$ and $y_x(a)$ have fixed values. Show that this variational problem leads to the Euler–Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y_{xx}}\right) = 0\,.$$

As part of your working, derive the boundary conditions

$$\frac{\partial f}{\partial y_x} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_{xx}}\right) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y_{xx}} = 0$$

that must be satisfied at $x = b$.

[8 marks]

(b) If the function f in part (a) has no explicit dependence on x, show (e.g., by direct differentiation) that the expression

$$f - y_x \frac{\partial f}{\partial y_x} + y_x \frac{d}{dx}\left(\frac{\partial f}{\partial y_{xx}}\right) - y_{xx}\frac{\partial f}{\partial y_{xx}}$$

is a first integral of the Euler–Lagrange equation in part (a).

[7 marks]

(c) A uniform metal bar of length L is clamped at one end, $x = 0$. The bar bends under its own weight, so that its vertical displacement at equilibrium is $y(x)$. The potential energy of the bar is given by

$$U[y] = \int_0^L \left[\tfrac{1}{2}\kappa y_{xx}^2 + \mu g y\right] dx\,,$$

where κ is an elastic constant, μ is the mass per unit length of the bar, and g is the acceleration due to gravity. Assuming that the bar is clamped so that $y(0) = 0$ and $y_x(0) = 0$, use your results from part (a) to find $y(x)$.

[10 marks]

END OF EXAMINATION PAPER