

Random Processes - Solutions 3

PC10471

- 3-1. The p.d.f. $f(t)$ is related to $P(t)$, the survival probability by

$$f(t) = -\frac{dP}{dt} = \alpha(t) P(t).$$

P is strictly positive, so $f \geq 0$ provided $\alpha \geq 0$.

Normalization:

$$\int_0^T f(t) dt = - \int_0^T \frac{dP}{dt} dt = -[P(t)]_0^T.$$

$$= P(0) - P(T).$$

$$\text{Now, } P(0) = e^0 = 1, \text{ so } \int_0^T f(t) dt = 1 - P(T). \\ = 1 - e^{-\int_0^T \alpha(t) dt}.$$

The last expression tends to 1 for large T only if $\int_0^T \alpha(t) dt$ becomes arbitrarily large; i.e., if $\int_0^T \alpha(t) dt \rightarrow \infty$ for $T \rightarrow \infty$.

- 3-2. Given:

$$\alpha(t) = \begin{cases} 0 & \text{for } t < 40 \text{ yr.} \\ \epsilon(t-40) & \text{for } t \geq 40 \text{ yr.} \end{cases}$$

with $\epsilon = 2.5 \times 10^{-3} \text{ yr}^{-2}$.

As above,

$$f(t) = -\frac{dP}{dt} = \alpha P \quad \text{①}$$

$$\Rightarrow \frac{df}{dt} = \frac{d}{dt}(\alpha P) = \frac{d\alpha}{dt} P + \alpha \frac{dP}{dt} \\ = \frac{d\alpha}{dt} P - \alpha^2 P \text{ after using ①.}$$

The smoker is most likely to become ill at time $t = \hat{t}$ when $f(t)$ has a maximum.

$$\frac{df}{dt}|_{\hat{t}} = 0, \Rightarrow \alpha'(\hat{t}) - \alpha^2(\hat{t}) = 0.$$

Although the last equation is satisfied for any $\hat{t} \leq 40$, $f = \alpha P = 0$ for these times, so f cannot be a maximum.

The non-trivial solution is for $\hat{t} > 40$:

$$\alpha'(\hat{t}) = \epsilon$$

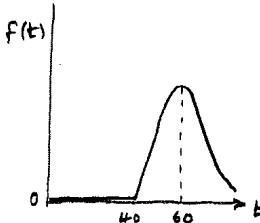
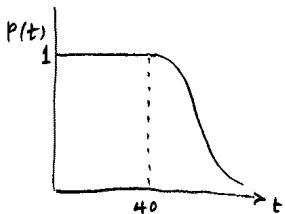
$$= (\alpha(\hat{t}))^2 = \epsilon^2 (\hat{t} - 40)^2$$

$$\Rightarrow (\hat{t} - 40)^2 = \frac{1}{\epsilon}$$

$$\Rightarrow \hat{t} = 40 + \frac{1}{\sqrt{\epsilon}} = 40 + \frac{1}{(2.5 \times 10^{-3})^{1/2}}$$

$$= 40 + (400)^{1/2} = 40 + 20 = \underline{\underline{60 \text{ yr.}}}$$

Sketch: $\int_0^t \alpha(t') dt' = \begin{cases} 0 & \text{for } t < 40 \\ \frac{1}{2} \epsilon(t-40)^2 & \text{for } t \geq 40 \end{cases}$



3-3. In this problem, $K(z)$ is the "hazard rate" (for absorption) which depends on position (rather than time). If $P(z)$ is the probability of survival to height z ,

$$P(z + \Delta z) = P(z) \times \underbrace{(1 - K(z)\Delta z)}_{\substack{\text{prob. of} \\ \text{survival to } z}} \underbrace{\Delta z}_{\substack{\text{prob. of not being} \\ \text{absorbed in interval} \\ [z, z + \Delta z], given \\ \text{survival to } z.}}$$

$$= P(z) - K(z) P(z) \Delta z$$

$$\Rightarrow \frac{P(z + \Delta z) - P(z)}{\Delta z} = -K(z) P(z)$$

For $\Delta z \rightarrow 0$, left-hand side becomes $\frac{dP}{dz}$
(definition of a derivative!), so

$$\frac{dP}{dz} = -K(z) P(z) \Rightarrow P(z) = e^{-\int_0^z K(z') dz'}.$$

With the given form for $K(z)$,

$$\begin{aligned} \int_0^z K(z') dz' &= \frac{1}{4} \int_0^z e^{-z'/8} dz' \\ &= \frac{1}{4} [-8e^{-z'/8}]_0^z = 2(1 - e^{-z/8}). \end{aligned}$$

$$\Rightarrow P(z) = e^{-2(1 - e^{-z/8})}$$

$\rightarrow e^{-2}$ for $z \gg 8$ km

$= 0.135$, i.e., a 13.5% probability
that the photon escaped.

3-4. Meteor shower, rate of fall $\alpha = 30 \text{ hr}^{-1}$. Recognize this as a typical application of the Poisson distribution.

Expected # in 10 min [parameter λ of Poisson dist.]

$$\begin{aligned} &= \alpha T \\ &= 30 \times \frac{10}{60} = \underline{\underline{5}}. \end{aligned}$$

$$\begin{aligned} \text{Probability of } k \text{ meteors} &= P_k = \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \frac{5^k}{k!} e^{-5}. \end{aligned}$$

$$(a) P_0 = e^{-5} = \underline{\underline{0.067}},$$

$$(b) P_5 = \frac{5^5}{5!} e^{-5} = \underline{\underline{0.175}}.$$

3-5 Radioactive decay, $\alpha = 3.2 \text{ s}^{-1}$.

Time interval $T = 1 \text{ s}$.

of counts in time T again follows the Poisson distribution with parameter (mean)

$$\lambda = \alpha T = 3.2.$$

$$\begin{aligned} P(\# \leq 2) &= P_0 + P_1 + P_2 = \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!}\right) e^{-\lambda} \\ &= \underline{\underline{0.380}}. \end{aligned}$$

3.6 Let $k = \#$ of electrons emitted in time t

$Q = ek$ is the charge transferred

$I = \frac{Q}{t} = \frac{ek}{t}$ is the current, averaged over time t .

Note that I is a random variable, because k follows the Poisson distribution.

$$\text{Mean: } \bar{I} = \frac{e}{t} \langle k \rangle^{\infty} \text{ (given)} \\ = er. \quad \textcircled{1}$$

$$\text{Variance: } \sigma_I^2 = \frac{e^2}{t^2} \sigma_k^2 = \frac{e^2 r}{t^2} \times (\bar{r})$$

Since $\sigma_k^2 = \langle k \rangle$ for a Poisson random var.

$$\text{Hence } \sigma_I^2 = \frac{e^2 r}{t^2} = \frac{e \bar{I}}{t} \text{ using } \textcircled{1}$$

$$\Rightarrow \sigma_I = \left(\frac{e \bar{I}}{t} \right)^{1/2}$$

i.e., the fluctuations in I become relatively more important as the interval of time decreased. (This is obvious because charge is transferred in discrete units, e ; and for a given \bar{I} , fewer electrons will be transferred in a shorter time.)

3.7 $\bar{b} = \text{average } \# \text{ of births per day.}$

Births are [presumably!] independent of one another, so the # b in one day should follow a Poisson distribution with mean \bar{b} .

For the conditional probability $P(b|b \geq 1)$ use the definition $P(X|Y) = P(X \cap Y)/P(Y)$.

$$\text{Here } P(Y) \text{ is } P(b \geq 1) = 1 - P(b=0) \\ = 1 - e^{-\bar{b}},$$

and $P(b \cap b \geq 1)$ is simply the Poisson P_b for $b \neq 0$. Hence $P(b|b \geq 1) = P_b/(1 - e^{-\bar{b}})$.

Expectation value:

$$\langle b \rangle|_{b \neq 0} = \sum_{b=1}^{\infty} b P(b|b \geq 1)$$

$$= \frac{1}{1 - e^{-\bar{b}}} \sum_{b=0}^{\infty} b P_b = \frac{\bar{b}}{1 - e^{-\bar{b}}}$$

can include 0, because that term is zero!

$$\text{For } \bar{b} \ll 1, \langle b \rangle|_{b \neq 0} \approx \frac{\bar{b}}{1 - (1 - \bar{b})} = \frac{\bar{b}}{\bar{b}} = 1$$

[obvious, since the prob. of 2 or more births is very small in this limit]

$$\text{For } \bar{b} \gg 1, \langle b \rangle|_{b \neq 0} \approx \bar{b} \quad (e^{-\bar{b}} \ll 1 \text{ here})$$

[unsurprising, since the probability of no births is $e^{-\bar{b}}$, which is very small in this limit.]

3-8.

$$\begin{array}{c} \text{# of stars per sq. deg.} \\ \uparrow \\ \boxed{A} \quad \begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix} \quad \downarrow \\ 0.5^\circ \quad \quad \quad = \frac{1500}{41000} \\ \leftarrow 0.5^\circ \rightarrow \quad \quad \quad = 3.66 \times 10^{-2} \text{ on average.} \end{array}$$

Mean # in 0.25 sq.deg. (A)

$$= 0.25 \times 3.66 \times 10^{-2}$$

$$= 9.15 \times 10^{-3} \equiv \lambda$$

Which we'll take as the parameter of a Poisson dist., assuming the placement of stars in sky to be completely random.

$$\text{Prob. of 6 stars in } A: P_6 = \frac{\lambda^6}{6!} e^{-\lambda} = 8.06 \times 10^{-16}$$

$$\text{Now, there are } \frac{41000}{A} = 164000$$

Such areas in the sky [what about overlap?? ignore that...], and the probability that none of them has 6 stars is

$$(1 - P_6)^{164000} \approx 1 - 164000 \times P_6$$

Here we keep just the 1st 2 terms of the binomial expansion $(1-x)^N = 1 - Nx + \frac{N(N-1)}{2}x^2 \dots$

because $x = P_6$ is so tiny. Hence prob. of 1 or more 6-star clusters is

$$1 - (1 - P_6)^{164000} \approx 164000 P_6$$

$$= 1.32 \times 10^{-10}$$

Conclusion? The Pleiades are not a grouping that arose by chance...