

PROPERTIES OF POISSON DISTRIBUTION

[SEE PLOTS of $P_k = \frac{\lambda^k}{k!} e^{-\lambda}$]

SMALL λ , < 1 : P_k decreases rapidly with increasing k . Largest is P_0 .

INTERMEDIATE λ : P_k rises to MAXIMUM, then falls. To find position of max, consider the ratio

$$\frac{P_k}{P_{k-1}} = \frac{\lambda^k e^{-\lambda}/k!}{\lambda^{k-1} e^{-\lambda}/(k-1)!} = \lambda \frac{(k-1)!}{k!} = \frac{\lambda}{k}$$

so that $P_k > P_{k-1}$ if $k < \lambda$,

$P_k < P_{k-1}$ if $k > \lambda$

[and $P_k = P_{k-1}$ if $k = \lambda$ (unusual)];

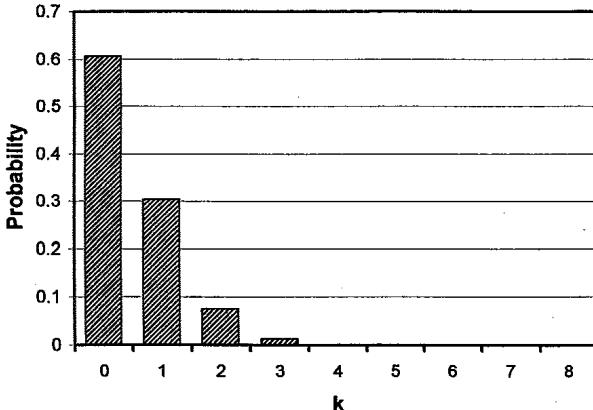
i.e. the MAX occurs for $k = \text{Int}(\lambda)$
[integer part of λ , "rounded down"].

LARGE λ : P_k becomes approx. GAUSSIAN with mean λ and s.d. $\sqrt{\lambda}$:

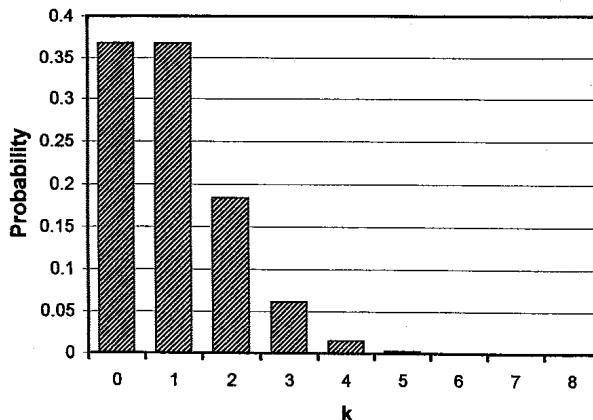
$$P_k \approx \frac{e^{-(k-\lambda)^2/2\lambda}}{\sqrt{2\pi\lambda}}$$

[Might return to this in another lecture]

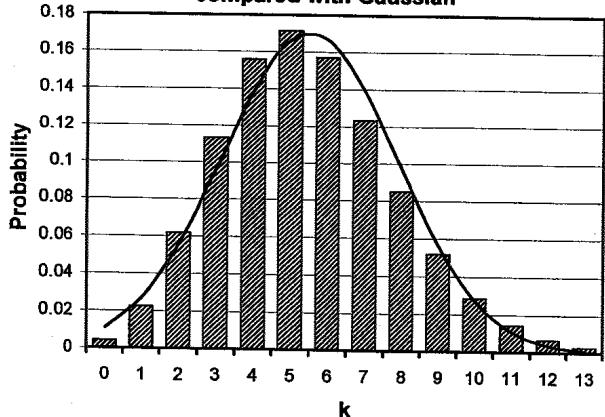
Poisson distribution, mean=0.5



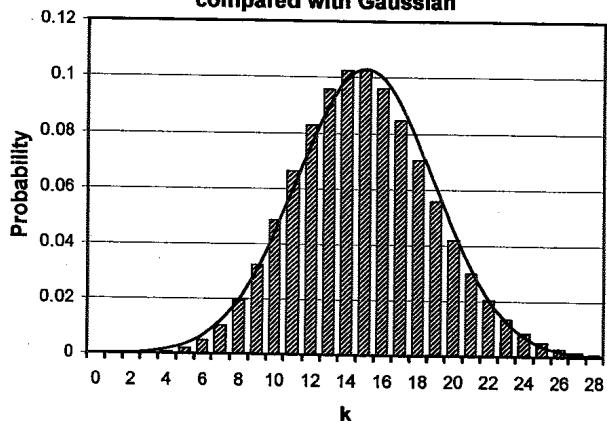
Poisson distribution, mean=1.0



Poisson distribution, mean=5.5,
compared with Gaussian



Poisson distribution, mean=15.0,
compared with Gaussian



[POISSON DIST., CONTINUED]

Quite often need conditional probabilities based on an underlying Poisson distribution. For example, a particle detector triggers if at least one ptc1 arrives; the fire engines are called out only if there is a fire; etc...

Use the fundamental

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{1 - P(\bar{B})}$$

which for this case is

$$\begin{aligned} P(k|k \neq 0) &= \frac{P_k}{P(k \neq 0)} = \frac{P_k}{1 - P_0} \\ &= \frac{\lambda^k e^{-\lambda}}{k! 1 - e^{-\lambda}} \end{aligned}$$

Expectation value of k , given $k \neq 0$, would be

$$\langle k \rangle_{k \neq 0} = \sum_{k=1}^{\infty} k P(k|k \neq 0)$$

[See Problem 3-7]

SUM OF POISSON RANDOM VARIABLES

Suppose ℓ and m are Poisson RVs with means λ and μ , respectively [e.g., ℓ might be # of hurricanes and m the # of tsunamis]. What's the distribution for their sum $s = \ell + m$? [s = # of disasters, i.e. hurricanes + tsunamis]

s may be made up in various ways —

$$(\ell, m) : (s, 0), (s-1, 1), \dots (s-k, k), \dots (0, s)$$

— all mutually exclusive possibilities.

Probability of scenario $\ell = s - k$, $m = k$

is $\left\{ \frac{\lambda^{s-k}}{(s-k)!} e^{-\lambda} \right\} \times \left\{ \frac{\mu^k}{k!} e^{-\mu} \right\}$

to be summed over the possible $k = 0, 1, \dots s$:

$$\begin{aligned} P_s &= \underbrace{\left[\sum_{k=0}^s \frac{\lambda^{s-k}}{(s-k)!} \frac{\mu^k}{k!} \right]}_{S!} \frac{e^{-(\lambda+\mu)}}{S!} \\ &= (\lambda+\mu)^s \frac{e^{-(\lambda+\mu)}}{S!} \quad [\text{USING BINOMIAL THEOREM}] \end{aligned}$$

i.e., s is ALSO a Poisson RV, whose mean $\langle s \rangle = \lambda + \mu$.

[SUM OF POISSON RVs]

ALTERNATIVE ARGUMENT: Think of ℓ and m as being the number of Geiger-counter "clicks" in time intervals λ and μ , for a mean count rate $\alpha = 1.0$. [These are Poisson RVs with means λ and μ .]

But $s = \ell + m$ would then be the total counts in time $\lambda + \mu$, which is a Poisson RV with mean $\langle s \rangle = \lambda + \mu$.

EASY EXTENSION: If $\ell_1, \ell_2, \dots \ell_n$ are Poisson-distributed RVs with means $\lambda_1, \lambda_2, \dots \lambda_n$, respectively, their sum $s = \sum_i \ell_i$ is a Poisson RV with mean $\langle s \rangle = \sum_i \lambda_i$.

[SUM OF POISSON RVs]

EXTENSION TO VARIABLE $\alpha(t)$

Although we derived the Poisson distribution for the case of constant hazard rate α , the result on "sums of Poisson RVs" makes extension to $\alpha(t)$ easy:

of counts in interval $[s, s + \Delta s]$
is Poisson RV with mean $\alpha(s) \Delta s$

\Rightarrow TOTAL counts in $[0, t]$ is also
Poisson RV with mean

$$\lambda = \sum_{\substack{\text{all} \\ \text{intervals } \Delta s}} \alpha(s) \Delta s \rightarrow \int_0^t \alpha(s) ds.$$

In fact, this is often used implicitly:
e.g. # of hurricanes per year will be
treated as Poisson RV, even though the
hazard rate is non-constant [hurricanes
happen mainly in the hurricane season]. But
we'd have to be explicit about it if
considering only part of the year.

We'd also need to be explicit about $\alpha(t)$
for RA source : $\alpha(t) = \alpha_0 e^{-t/T}$
unless $T \gg$ time-scale of experiment.