Random Processes: Examples 4 — M J Godfrey

Key: **E**asy; **M**oderate; **D**ifficult; **O**ptional

1. [E] An ABM defence system is 99% effective in intercepting incoming ballistic missiles. Calculate the probability that it will successfully destroy 100 incoming missiles.

   How many missiles should the enemy launch to have a better than 50% chance of at least one missile penetrating the defence system?

2. The back-up electrical power system for a certain scientific laboratory consists of 5 independent diesel-powered generators, each of which functions with a probability of 90% when called upon. Some redundancy has been built in to the system, such that adequate power could be provided by any 3 of the generators. Calculate the probability that sufficient power will be available from the back-up system during a power cut.

3. [M] In the UK National Lottery draw discussed in Problem 2.3, you calculated the probabilities P_r that a player chooses r of the six numbers correctly. The values found were $P_3 = 0.0176$, $P_4 = 9.69 \times 10^{-4}$, $P_5 = 1.84 \times 10^{-5}$ and $P_6 = 7.15 \times 10^{-8}$. Given that 32, 740, 471 tickets were sold, the *expected* numbers of winning tickets for each value of r were then: $\langle N_3 \rangle = 577,883$; $\langle N_4 \rangle = 31,713$; $\langle N_5 \rangle = 604$; and $\langle N_6 \rangle = 2.3$. At first sight, these numbers seemed to be roughly in line with the observed results: $N_3 = 582,729$; $N_4 = 29,988$; $N_5 = 423$; and $N_6 = 2$.

   Use the binomial distribution to calculate the standard deviations of the quantities N_r for $r = 3$ to 5, and re-assess the level of agreement found. [Keep your analysis simple: knowledge of the χ^2 test is *not* required for this course!]

   Re-examine the assumptions that led to the calculated P_r and suggest a possible reason for large discrepancies between the observed and calculated values of N_r. Depending on your conclusion, either:

   (i) write a letter to the *Guardian* to express your outrage at uncovering a capitalist conspiracy to fix the results of the National Lottery, or

   (ii) devise a cunning plan to reduce the risk of having to share the prize money next time *you* win the jackpot.

4. [EM] A container of volume V contains N molecules of a gas at low density. The probability that a given molecule is found in a volume v inside the container is $p = v/V$, and different molecules may occupy v independently of one another (i.e., the effects of their interaction with one another can be neglected at low density). Explain why the probability that k of the N molecules are inside v is given by

$$P_k = \frac{N!}{k!\,(N-k)!}\,p^k(1-p)^{N-k}.$$

   Write down expressions for the mean, $\langle k \rangle$, and standard deviation, σ_k, for the number of molecules in v. How do the results simplify when $V \gg v$, with the number-density N/V kept at a constant value, $\bar{n}$?

Even clear air scatters sunlight because there are significant variations in the numbers of molecules in volumes of air whose size is comparable with the wavelength of light. Without these small-scale density fluctuations, the sky would appear black.

Estimate the typical magnitude of the density fluctuations by calculating a numerical value for $\sigma_k/\langle k\rangle$ for a cube of side 500 nm in an ideal gas at temperature 273 K and pressure 100 kPa.

[M] How would the simple picture of light scattering used here help to explain why the sky is *blue* (rather than red, say), and why the sun appears *red* (rather than blue) when close to the horizon?

5. [EM] After an unusually arduous lecture on Random Processes, a student staggers out of the building and takes steps of equal length d up and down Brunswick Street. Each step is independent of the last and each is equally likely to be towards or away from Oxford Road.

   For a walk of 100 steps, calculate:

   (a) the *expected* displacement of the student up the road after the walk;

   (b) the root-mean-square distance from the starting point; and

   (c) the probability that the student is more than 10 steps closer to Oxford Road after the walk.

   [Use the Gaussian limit of the binomial distribution for part (c). The probability that the measured value of a Gaussian-distributed random variable lies within one standard deviation of its mean is 68.3% and the probability that it lies within two standard deviations of its mean is 95.4%.]

6. [E] A long, flexible chain molecule consists of $N+1$ chemical units connected by N chemical bonds of length a. If $\mathbf{r}_i$ is the position vector of the ith unit, we can write $\mathbf{r}_i - \mathbf{r}_{i-1} = \mathbf{a}_i$, where $|\mathbf{a}_i| = a$. By assuming that the bond directions are completely independent of one another [i.e., that $\langle \mathbf{a}_i \cdot \mathbf{a}_j\rangle = 0$ for $i \neq j$], show that the mean-squared end-to-end length of the molecule is

$$R^2 = \langle (\mathbf{r}_N - \mathbf{r}_0)^2\rangle = Na^2.$$

   [M] In a more realistic model we might suppose that the bond directions are correlated in a way which decreases with their separation along the molecule: $\langle \mathbf{a}_i \cdot \mathbf{a}_j\rangle = a^2\lambda^{|i-j|}$ for any i and j, where $0 < \lambda < 1$. You can verify that such a correlation tends to *increase* the end-to-end length of the molecule. First show that

$$\langle (\mathbf{r}_N - \mathbf{r}_0)^2\rangle = Na^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\langle \mathbf{a}_i \cdot \mathbf{a}_j\rangle = Na^2 + 2\sum_{i=1}^{N-1}\sum_{k=1}^{N-i} a^2\lambda^k,$$

   where, to obtain the last expression, the summation variable has been transformed using $k = j - i$. By approximating the sums in a way which is appropriate for large N [most of the bonds are far from the ends of the molecule], show that

$$R^2 = \langle (\mathbf{r}_N - \mathbf{r}_0)^2\rangle \simeq \left\{1 + \frac{2\lambda}{1-\lambda}\right\} Na^2 \quad \text{for} \quad N \gg 1.$$

   [O/D] For a slightly harder problem, evaluate the sums *exactly*, and check your final result for R^2 by comparing with the easy cases $N = 1$ and 2.