Random Processes: Examples 3 　　　 M J Godfrey

Key: **E**asy; **M**oderate; **D**ifficult; **O**ptional

1. [EM] In Lecture 10 we considered a generalization of the exponential probability distribution to problems where the hazard rate, $\alpha(t)$, was not constant. We found that the survival probability was given by

$$P(t) = e^{-\int_0^t \alpha(t')\,\mathrm{d}t'}.$$

Verify that the corresponding probability density function $f(t)$ has the right properties to be a p.d.f., namely,

$$f(t) \geq 0 \quad \text{and} \quad \int_0^\infty f(t)\,\mathrm{d}t = 1,$$

provided that $\alpha(t) \geq 0$ for all $t \geq 0$ and that

$$\int_0^t \alpha(t')\,\mathrm{d}t' \to \infty \quad \text{for} \quad t \to \infty.$$

Note that if the last condition is *not* satified, $P(t)$ is not necessarily incorrect: it just means that there is a non-zero probability of survival to indefinitely long times.

2. [EM] In a simplified model for the lung cancer hazard rate among smokers, the function $\alpha(t)$ in Q. 1 may be approximated by $\alpha(t) = 0$ for $t < 40$ years, and $\alpha(t) = \epsilon(t - 40)$ for $t \geq 40$ years, where $\epsilon = 2.5 \times 10^{-3}\,\mathrm{yr}^{-2}$. By following the method discussed in Lecture 10, show that a smoker is most likely to get cancer at the age of 60. [You should ignore all other hazards for the purposes of this problem.]

3. [EM] Microwave photons from a ground-based maser [a microwave laser] travel vertically upwards through the Earth's atmosphere. The probability that a photon is absorbed at a height between z and $z + \Delta z$ above the ground (assuming that it has got to height z) is $K(z)\,\Delta z$. The function $K(z)$ is proportional to the density of the atmosphere, and may be approximated by

$$K(z) = 0.25\,e^{-z/8},$$

where z is measured in km. Find an expression for the probability that the photon survives to height z without absorption. Hence show that a photon has a 13.5% chance of escaping from the Earth's atmosphere without being absorbed.

4. [E] The Leonid meteor shower of 2004 was unusually heavy during the night of November 18/19, with an average of 30 meteors seen per hour during the period of peak activity. What was the *expected* number of meteors seen during a ten-minute period of observation? Calculate the probability of observing (a) no meteors and (b) five meteors in a given ten-minute period.

5. [E] A feeble radioactive source emits 3.2 alpha particles per second on average. Calculate the probability that no more than two alpha particles are emitted in a one-second interval.

6. [E] The number of electrons emitted in time t from a heated metal filament follows a Poisson distribution with mean rt. Show that the mean and standard deviation of the *electric current* during an interval of time t can be expressed as

$$\bar{I} = er \qquad \text{and} \qquad \sigma_I = \left(e\bar{I}/t\right)^{1/2}.$$

7. [M] Let $\bar{b}$ be the average number of births per day in the maternity ward of a hospital. Show that the average number of births on days when there is at least one birth is $\bar{b}/(1 - \exp[-\bar{b}])$. [Start by working out the conditional probability that there are b births, given that there is at least one birth.] Show that this average is approximately 1 for the limiting case $\bar{b} \ll 1$, and $\bar{b}$ for the limiting case $\bar{b} \gg 1$. Give a qualitative explanation of the results in these limiting cases.

   [You may wish to use the expansion $e^x \simeq 1 + x$, valid for $x \ll 1$.]

8. [MD] The Pleiades star cluster is a conspicuous grouping of stars, six of which are easily visible to the naked eye. The cluster covers about 0.25 square degrees of the sky, by which is meant a region with angular dimensions such as $0.5^\circ \times 0.5^\circ$. There are approximately 1500 stars at least as bright as the Pleiades. Assuming that these are randomly distributed over the entire sky [an area of about $41,000$ square degrees], calculate the probability that a *given* region of 0.25 square degrees contains exactly six stars. Hence estimate the probability that at least *one* such cluster would appear somewhere in the sky. In the light of your last result, do you believe that the Pleiades star cluster is likely to have arisen by chance?