Random Processes: Examples 2 — M J Godfrey

Key: **E**asy; **M**oderate; **D**ifficult; **O**ptional

1. [E] A die is rolled repeatedly until a "1" is obtained. Find the probability that the "1" is obtained on the kth attempt. Calculate the expected number of attempts needed to obtain a "1". [Ans: 6]

2. [EM] In a simple model of α-decay of a radioactive nucleus, the α-particle makes repeated attempts to penetrate a potential barrier until it succeeds and escapes. If the probability of escape is p for any given attempt, work out a formula for the probability that the α-particle escapes on its kth attempt. Also find the expected number of attempts needed for the α-particle to escape.

   [Hint: Note the similarity between Q. 2 and Q. 1.]

3. [M] In a UK National Lottery draw, there are 49 numbered balls, 6 of which are selected at random. Players have to choose 6 numbers and may receive a prize if r of their choices match the balls drawn. [The *order* in which the balls are drawn or numbers selected is irrelevant, of course.]

   (a) What is the total number of ways of drawing 6 balls from 49?

   (b) In how many ways can the balls be drawn such that a given ticket has r winning numbers *and* $6 - r$ losing numbers?

   (c) Hence show that the probability that a ticket has r winning numbers is

   $$P_r = \frac{\binom{6}{r}\binom{43}{6-r}}{\binom{49}{6}}.$$

   (d) [O] Use the result from Examples 1, Q. 4 to show (without any numerical calculation) that the probabilities in (c) add up to unity.

   (e) On 26 September 2001, 32,740,471 national lottery tickets were bought. The numbers N_r of winning tickets with r choices correct were: $N_3 = 582,729$; $N_4 = 29,988$; $N_5 = 423$; and $N_6 = 2$. Check that these numbers are roughly consistent with your result from part (c).

   (f) [O] In answering part (e) you may have noticed that you needed to make an extra assumption [which is not simply that all possible draws are equally likely]. If you noticed the subtlety, explain why the assumption might well be wrong.

4. [E] Assume that the wingspan of the common pipistrelle bat is a Gaussian random variable with mean $\mu = 21\,\text{cm}$ and standard deviation $\sigma = 2\,\text{cm}$. What is the probability that a common pipistrelle has a wingspan greater than 25 cm?

   [Data: The probability that a Gaussian distributed random variable is within one standard deviation of its mean is 68% and the probability that it is within two standard deviations of its mean is 95%.]

5. [EM] Consider the decay of a radioactive nucleus again (Q. 2 above). Assume that escape attempts occur at short intervals Δt and that the probability of success $p = \lambda \Delta t$, where λ remains finite as Δt tends to zero.

   Work out an expression for the probability that the nucleus has not decayed before time t. Find the expected lifetime of the nucleus in terms of λ.

6. [M] Naturally occurring uranium occurs today as a mixture of two isotopes in the proportions 0.7% ^{235}U (half-life 0.7×10^9 years) and 99.3% ^{238}U (half-life 4.5×10^9 years). Calculate the proportions of these isotopes in natural uranium 1.8×10^9 years ago.