Random Processes: Examples 1 M J Godfrey

Key: **E**asy; **M**oderate; **D**ifficult; **O**ptional

1. [E] On 18 August 1913, black came up at the roulette wheel in the Casino in Monte Carlo 26 times in succession. Needless to say, the punters, seeing a run of black, started betting on red. They lost.

   On a European roulette wheel there are 37 numbers (0 to 36), of which 18 are black and 19 are non-black. For 26 spins of a roulette wheel, work out

   (a) the total number of possible outcomes

   (b) the number of outcomes in which all the numbers are black

   (c) the *a priori* probability of all the numbers being black, assuming that all the outcomes in (a) are equally likely.

2. [EM] A single suit of 13 cards is shuffled. By calculating the numbers of favourable configurations in each case, find the probability of the following results:

   (a) the top card is the ace.

   (b) the top card is *either* the ace *or* the king.

   (c) the bottom card is the king.

   (d) either the top card is the ace, or the bottom one is the king, or both. Why is the answer different from (b)?

   (e) the top card is the ace and the bottom one is the king. Why isn't the answer simply (a) × (c)?

3. [E] By writing down the binomial theorem for $(x + y)^n$ and making an appropriate choice for x and y, show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

   Alternatively, use a combinatorial argument to work out the total number of ways of selecting a committee of any size from a group of n people. [Hint: Each person can either be *in* the committee or *not* in it!]

   Also show that

$$\sum_{k=0}^{n} k \binom{n}{k} = n\, 2^{n-1}.$$

   [Either differentiate $(x+y)^n$ with respect to x, or think about the ways in which you can choose committees of size $k = 1, 2, \ldots, n$ and then a chairperson for the committee.]

4. [M] Show that

$$\binom{m+w}{n} = \binom{m}{0}\binom{w}{n} + \binom{m}{1}\binom{w}{n-1} + \ldots + \binom{m}{n}\binom{w}{0};$$

you can assume that $n \leq m$ and $n \leq w$.

Hint: Consider a group of m men and w women and work out the various ways in which a group of size n can be selected from them.

[MD/O] Alternatively, work out for yourself how to derive the above result from the binomial theorem.

5. [EM] Lord Yarborough (1809–1862) was prepared to bet £1000 to £1 against the occurrence of a bridge hand containing no card higher than 9. Calculate the probability of being dealt such a hand, and decide whether you would accept the bet. [A bridge hand is 13 cards dealt from a normal 52-card deck. Aces are high cards in bridge.]

6. [M] A lecturer provides answers to a set of problems. The probability that he actually knows the answer to (say) Q. 6 is $\frac{4}{5}$, and the probability that he has guessed is $\frac{1}{5}$. If he has guessed, there is a 1 in 3 chance that he has guessed correctly. Given that the answer actually *is* correct, find the probability that the lecturer really knew the answer.

7. [M] A certain bloodtest for rheumatoid arthritis (RA) is 95% effective in identifying the disease in 20-year-olds who actually have the disease, but there is a 1% chance that even a healthy individual will test positive for RA.

   Assuming that 0.5% of 20-year-olds have RA, show that the probability of a person of this age having the disease is 32%, if they tested positive for it.

8. [M] A beam of mesons is composed of 90% pions and 10% kaons. A pion detector is 95% accurate in giving a positive signal for a pion, but has a 6% chance of falsely detecting a pion when a kaon arrives.

   Show that when a particle is identified as a pion by the detector, the chance that this particle really is a pion is 99.3%.