PC4902: Elements of QMBT, Pt 2 Problems 4

Key: **E**asy; **M**oderate; **D**ifficult

1. [EM] The Holstein–Primakoff representation of spins-S in terms of Bose operators $\hat{a}$ and $\hat{a}^\dagger$ has been discussed in lectures. The following approximation to it has also been used in our treatment of spin waves:

$$\hat{S}^+ \simeq \sqrt{2S}\,\hat{a}^\dagger, \quad \hat{S}^- \simeq \sqrt{2S}\,\hat{a}, \quad \hat{S}^z = -S + \hat{a}^\dagger\hat{a}.$$

By evaluating the commutators of the spin components, discuss the conditions under which you would expect this to become a *good* approximation.

2. [M] Exam 2002–3, adapted: part (b) extended; part (c) omitted.

   (a) In a one-dimensional Heisenberg ferromagnet of spins of magnitude $S = \frac{1}{2}$, the Hamiltonian operator takes the form

$$\hat{H} = -|J| \sum_n \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1}\,,$$

   where J is the exchange coupling constant, and the summation over n includes all N spins in the chain. Show that a state $\psi^{(0)}$ with all the spins aligned is an eigenstate of $\hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1}$, and find the corresponding eigenvalue. Hence show that $\psi^{(0)}$ is an eigenstate of $\hat{H}$ with energy eigenvalue $E^{(0)} = -\frac{1}{4}N|J|$. [Neglect end effects, or suppose that the chain forms a closed loop.] You may assume later that $\psi^{(0)}$ is a ground state of the system.

   (b) If in $\psi^{(0)}$ all the spins are aligned in the negative z direction, a state with one spin deviation at lattice site n may be defined by $\psi_n \equiv \hat{S}_n^+\psi^{(0)}$. Show that

$$\hat{H}\psi_n = |J|\big(\psi_n - \tfrac{1}{2}\psi_{n+1} - \tfrac{1}{2}\psi_{n-1}\big) + E^{(0)}\psi_n\,.$$

   Hence show that a Bloch state constructed from the ψ_n has energy

$$\epsilon_k = |J|(1 - \cos ka)$$

   in excess of $E^{(0)}$, where k is the wave vector and a is the lattice spacing. [Hint: Construct this state in the same way that a Bloch wave function is built up from atomic wave functions in the tight-binding approximation.] What is the physical significance of this solution of Schrödinger's equation?

3. [M] Apply the equation of motion method to the problem solved in Q. 2 (b) above. You will need to find the commutator of $\hat{S}_n^+$ with the Hamiltonian and work out what linear combinations of the $\hat{S}_n^+$ will satisfy the "equation of motion" for the creation operator of an excitation: you should be able to make an educated guess for the form that these combinations take, given the approach used above. You will need to use the fact that the operator will be applied only to the spin-aligned ground state.

   The final result for ϵ_k in the last two questions is the same as would be obtained by the method used in the lecture, where we made the lowest-order approximation to the Holstein–Primakoff transformation. Can you give a *simple* explanation of why the "approximation" gives the exact result in this case?