PC4902: Elements of QMBT, Pt 2 — Problems 0

All of the problems on this sheet can be regarded as *optional*. They are provided to help you test and extend your understanding of second quantization.

1. Fermion creation and annihilation operators have been defined by their effects on the basis states of the occupation number representation:

$$\begin{aligned} \hat{c}_i^\dagger |n_1, n_2, \ldots, 0_i, \ldots\rangle &= (-1)^{\Sigma_i} |n_1, n_2, \ldots, 1_i, \ldots\rangle \\ \hat{c}_i |n_1, n_2, \ldots, 1_i, \ldots\rangle &= (-1)^{\Sigma_i} |n_1, n_2, \ldots, 0_i, \ldots\rangle, \end{aligned}$$

where $\Sigma_i = \sum_{j=1}^{i-1} n_j$. We should supplement these definitions with conditions that enforce the constraint $n_i = 0$ or 1 only:

$$\hat{c}_i^\dagger |n_1, n_2, \ldots, 1_i, \ldots\rangle = \hat{c}_i |n_1, n_2, \ldots, 0_i, \ldots\rangle = 0$$

Show to your own satisfaction that the above definitions lead to the anticommutation relations

$$\hat{c}_i \hat{c}_j + \hat{c}_j \hat{c}_i = \hat{c}_i^\dagger \hat{c}_j^\dagger + \hat{c}_j^\dagger \hat{c}_i^\dagger = 0 \quad \text{and} \quad \hat{c}_i \hat{c}_j^\dagger + \hat{c}_j^\dagger \hat{c}_i = \delta_{ij}.$$

2. Consider the state

$$|\phi\rangle = |1, 1, 1, 1, 1, 0, 0, \ldots\rangle = \hat{c}_1^\dagger \hat{c}_2^\dagger \hat{c}_3^\dagger \hat{c}_4^\dagger \hat{c}_5^\dagger |\rangle,$$

where $|\rangle = |0, 0, 0, 0, 0, 0, 0, \ldots\rangle$ is the *vacuum state*, which is annihilated by all $\hat{c}_i$. [The vacuum state is sometimes denoted by $|\text{vac}\rangle$ or $|0\rangle$.] Evaluate $\hat{c}_3^\dagger \hat{c}_6 \hat{c}_4 \hat{c}_6^\dagger \hat{c}_3 |\phi\rangle$.

Write down an expression for $|1, 1, 0, 1, 1, 0, 1, 0, \ldots\rangle$ obtained by applying creation and annihilation operators to $|\phi\rangle$. The result is unique only if you aim to *minimize* the number of creation and annihilation operators used. Why?

3. For either bosons or fermions, evaluate the expectation value of the operator for the total number of particles, $\hat{N} = \sum_i \hat{c}_i^\dagger \hat{c}_i$, for the state

$$|\phi\rangle = A\, \hat{c}_1^\dagger |\rangle + B\, \hat{c}_1^\dagger \hat{c}_2^\dagger \hat{c}_4^\dagger |\rangle,$$

where A and B are constants. Note that $|\phi\rangle$ is a state with an indefinite number of particles—it is *not* an eigenstate of $\hat{N}$. In fact, what state *do* you get when you apply $\hat{N}$ to $|\phi\rangle$?

4. Consider the Bose field operator

$$\hat{\psi}(\mathbf{r}) = \sum_i \hat{a}_i u_i(\mathbf{r}).$$

Show that if the "expansion" for $\hat{\psi}$ is made using a different set of basis functions $\{v_i\}$,

$$\hat{\psi}(\mathbf{r}) = \sum_i \hat{b}_i v_i(\mathbf{r}),$$

the coefficients $\hat{b}_i$ are related to the original $\hat{a}_i$ by

$$\hat{b}_i = \sum_j \langle v_i | u_j \rangle \hat{a}_j.$$

Use the last expression (and its Hermitian conjugate) to show explicitly that the operators $\hat{b}_i$ and $\hat{b}_i^\dagger$ obey the usual Bose commutation relations, provided the original $\hat{a}_i$ and $\hat{a}_i^\dagger$ obey them.

One conclusion from this exercise is that the field operator $\hat{\psi}(\mathbf{r})$ is independent of the set of single-particle basis functions used to make the expansion. We had assumed this without comment when we deduced the second-quantized forms for a one-particle operator,

$$\hat{F}_1 = \int \hat{\psi}^\dagger(\mathbf{r}) \hat{f} \hat{\psi}(\mathbf{r}) \, \mathrm{d}^3\mathbf{r} = \sum_{i,j} \langle i | \hat{f} | j \rangle \hat{a}_i^\dagger \hat{a}_j,$$

having justified this only for the basis in which $\hat{f}$ was diagonal.

5. Show that if $\hat{a}^\dagger$ is a Bose or Fermi creation operator and $\hat{n} = \hat{a}^\dagger \hat{a}$, then

$$F(\hat{n}) \, \hat{a}^\dagger = \hat{a}^\dagger F(\hat{n} + 1),$$

where F is *any* function defined on the set of integers. [By $F(\hat{n})$ we mean an operator which is diagonal in the basis in which $\hat{n}$ is diagonal, and for which $F(\hat{n})|m\rangle = F(m)|m\rangle$, if $\hat{n}|m\rangle = m|m\rangle$.]

Use the above result to show that

$$e^{-\lambda \hat{n}} \hat{a}^\dagger e^{\lambda \hat{n}} = e^{-\lambda} \hat{a}^\dagger,$$

where λ is a constant.